International Mathematical Olympiad

Preliminary Selection Contest 2011 — Hong Kong

Outline of Solutions

Answers:

1.	9900	**2.**	7046	**3.**	84	**4.**	1211101
5.	100	**6.**	$\frac{2}{9}$	**7.**	$\frac{1}{33}$	**8.**	$\frac{11}{16}$
9.	$\frac{4\sqrt{3}-3}{4}$	**10.**	6	**11.**	$\frac{16}{5}$	**12.**	$\frac{3}{5}$
13.	$\frac{1}{8}$	**14.**	1005	**15.**	33	**16.**	–3
17.	22°	**18.**	1764	**19.**	963090	**20.**	1729

Solutions:

1. Such fractions include $10\frac{1}{3}$, $10\frac{2}{3}$, $11\frac{1}{3}$, $11\frac{2}{3}$, …, $99\frac{1}{3}$, $99\frac{2}{3}$. They can be grouped into 90 pairs, each with sum 110 (i.e. $10\frac{1}{3}+99\frac{2}{3}=110$, $10\frac{2}{3}+99\frac{1}{3}=110$, $11\frac{1}{3}+98\frac{2}{3}=110$ and so on). Hence the answer is $110\times 90=9900$.

2. The sum of 20 consecutive positive integers has unit digit equal to that of $2(0+1+\cdots+9)$, which is 0. Hence $f(n+20)=f(n)$ for all n. By direct computation, we have

$$\begin{aligned} f(1)+f(2)+\cdots+f(20) &= 1+3+6+0+5+1+8+6+5+5+6+8+1+5+0+6+3+1+0 \\ &= 70 \end{aligned}$$

Similarly, we have $f(21)+f(22)+\cdots+f(40)=70$, $f(41)+f(42)+\cdots+f(60)=70$ and so on, until $f(1981)+f(1982)+\cdots+f(2000)=70$. Altogether, we have 100 groups of 70. Finally, since

$$\begin{aligned} f(2001)+f(2002)+\cdots+f(2011) &= f(1)+f(2)+\cdots+f(11) \\ &= 1+3+6+0+5+1+8+6+5+5+6 \\ &= 46 \end{aligned}$$

the answer is $70\times 100+46=7046$.

3. We must have $\frac{11k}{4}-199\geq 0$ and so $k\geq 73$. On the other hand, the product of digits of k must not exceed k (to see this, suppose k is an n-digit number with leftmost digit b; then $k\geq b\times 10^{n-1}$ but the product of digits of k is at most $k\times 9^{b-1}$), so we have $\frac{11k}{4}-199\leq k$ and thus $k\leq 113$.

   Since $\frac{11k}{4}-199$, being the product of digits of k, must be an integer, k is divisible by 4. Hence the product of digits of k is also even, i.e. $\frac{11k}{4}$ is odd. Therefore $k\equiv 4 \pmod 8$, so the possible values of k include 76, 84, 92, 100 and 108. It is easy to check that only 84 works.

4. Note that $14641=11^4$ and $121=11^2$. Hence

$$\begin{aligned} 1464101210001 &= 14641\times 10^8+121\times 10^4+1 \\ &= 1100^4+1100^2+1 \\ &= 1100^4+2\times 1100^2+1-1100^2 \\ &= (1100^2+1)^2-1100^2 \\ &= (1100^2+1+1100)(1100^2+1-1100) \\ &= 1211101\times 1208901 \end{aligned}$$

   and so 1211101 is a possible answer.

   Remark. It can be checked that the answer is unique.

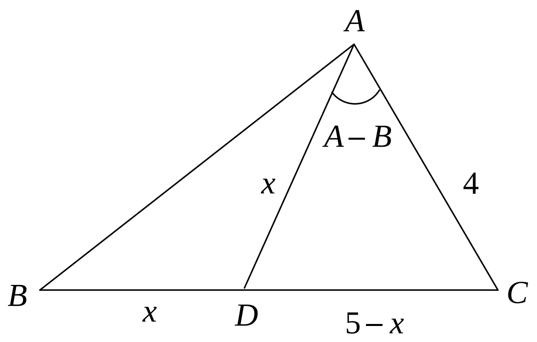
5. Note that there are 9 different letters, with 'I' and 'M' occurring twice. If the three letters are distinct, there are $C_3^9=84$ combinations. Otherwise, there are either two 'I's or two 'M's, plus one of the remaining 8 letters. There are $2\times 8=16$ such combinations. Hence the answer is $84+16=100$.

6. We have $y=a\left(x-\frac{1}{4}\right)^2-\frac{9}{8}$. Note that $a+b+c$ is equal to the value of y when $x=1$, i.e. $a+b+c=a\left(1-\frac{1}{4}\right)^2-\frac{9}{8}=\frac{9a-18}{16}$. Hence we need to look for the smallest positive a for

which $9a-18$ is divisible by 16. This corresponds to $9a-18=-16$, or $a=\frac{2}{9}$.

7. The four points form a rectangle if and only they are two pairs of diametrically opposite points. Hence among the $C_4^{12}=495$ possible outcomes, there are $C_2^6=15$ favourable outcomes (by choosing 2 out of the 6 pairs of diametrically opposite points). Therefore the answer is $\frac{15}{495}=\frac{1}{33}$.

8. Let D be a point on BC such that $DA=DB$. Then $\cos\angle DAC=\cos(A-B)=\frac{7}{8}$. Let $DB=DA=x$. Then $DC=5-x$ and applying cosine law in ΔDAC gives



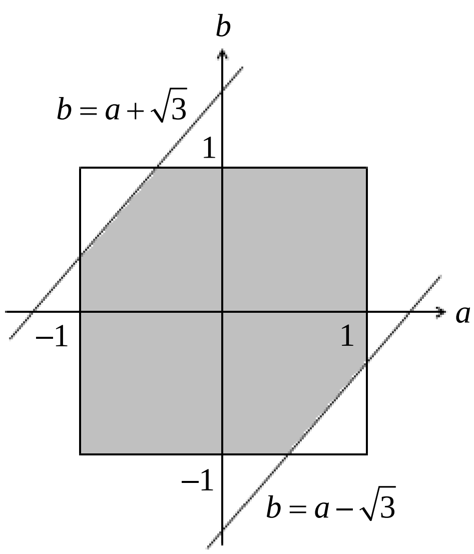
$$(5-x)^2=x^2+4^2-2(x)(4)\left(\frac{7}{8}\right),$$

or $x=3$, i.e. $DA=3$ and $DC=2$. Applying cosine law in ΔDAC again, we have

$$\cos C=\frac{4^2+2^2-3^2}{2(4)(2)}=\frac{11}{16}.$$

9. Let $P=(0, a)$ and $Q=(1, b)$, where each of a and b is randomly chosen from –1 to 1. Since each circle has radius 1, the two circles intersect if and only if $2\ge PQ=\sqrt{(1-0)^2+(b-a)^2}$, or $-\sqrt{3}\le b-a\le\sqrt{3}$.

On the Cartesian plane (with axes labelled a and b), the set of possible outcomes is the square bounded by the lines $a=\pm 1$ and $b=\pm 1$. The set of favourable outcomes is the region of the square between the lines $b=a-\sqrt{3}$ and $b=a+\sqrt{3}$, as shown.

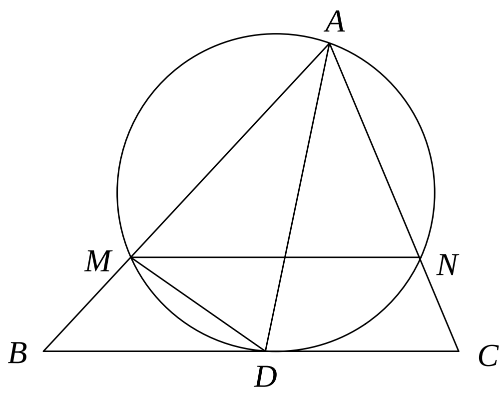


Consider the non-shaded triangular part of the square in the lower right hand corner. It is easy to find that the coordinates of the vertices of the triangle are $(1, 1-\sqrt{3})$, (1, –1) and $(\sqrt{3}-1, -1)$. Thus it is a right-angled isosceles triangle with leg $2-\sqrt{3}$ and whose area is $\frac{1}{2}\left(2-\sqrt{3}\right)^2=\frac{7-4\sqrt{3}}{2}$. The same is true for the non-shaded triangular part of the square in the upper left hand corner. Hence the required probability is $\dfrac{4-\left(7-4\sqrt{3}\right)}{4}=\dfrac{4\sqrt{3}-3}{4}$.

10. By the angle bisector theorem, $\dfrac{BD}{DC}=\dfrac{AB}{AC}=\dfrac{9}{7}$ and so $BD=8\times\dfrac{9}{9+7}=\dfrac{9}{2}$. By the power chord theorem, $BD^2=BM\times BA$ and so $BM=\dfrac{9}{4}$. Finally, MN is parallel to BC since

$$\angle BDM=\angle MAD=\angle DAN=\angle DMN.$$

Hence we have $\dfrac{AM}{AB}=\dfrac{MN}{BC}$, or $\dfrac{9-\frac{9}{4}}{9}=\dfrac{MN}{8}$, so that $MN=6$.



11. Since the lengths of the altitudes are inversely proportional to the side lengths, the sides of the triangle are in ratio $\dfrac{1}{3}:\dfrac{1}{4}:\dfrac{1}{6}=4:3:2$. Let the side lengths be $4k$, $3k$, $2k$. Then the triangle has perimeter $9k$ and Heron's formula asserts that it has area

$$k^2\sqrt{4.5(4.5-4)(4.5-3)(4.5-2)}=\frac{3\sqrt{15}}{4}k^2.$$

Since the in-radius of a triangle is equal to twice the area divided by the perimeter, we have

$$2\times\frac{3\sqrt{15}}{4}k^2\div 9k=1,$$

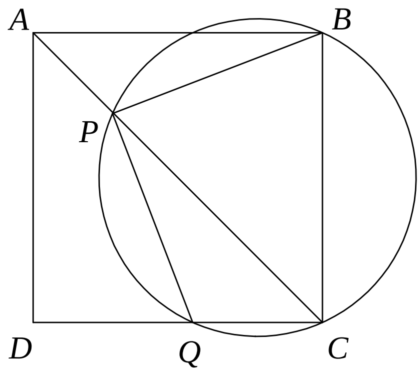
which gives $k=\dfrac{6}{\sqrt{15}}$. Let θ be the angle opposite the shortest side and R be the radius of the circumcircle. Then the sine law and cosine law give

$$2R=\frac{2k}{\sin\theta}=\frac{2k}{\sqrt{1-\cos^2\theta}}=\frac{2k}{\sqrt{1-\left(\dfrac{3^2+4^2-2^2}{2\cdot 3\cdot 4}\right)^2}}$$

and hence $R=\dfrac{16}{5}$.

12. Since $BPQC$ is a cyclic quadrilateral with $\angle C = 90°$, we have $\angle BPQ = 90°$. Also, $PB = PQ$ since $\angle PCB = \angle PCQ = 45°$. By Ptolemy's Theorem, we have

$$PB \times QC + PQ \times BC = PC \times BQ.$$



Since $BQ = \sqrt{2}PB = \sqrt{2}PQ$ and $BC = 1$, this simplifies to $QC + 1 = \sqrt{2}PC$. Let $CQ = y$. Then $PC = \frac{y+1}{\sqrt{2}}$ and considering the area of ΔCPQ gives $\frac{1}{2}\left(\frac{y+1}{\sqrt{2}}\right)y \sin 45° = \frac{6}{25}$, or $(5y+8)(5y-3) = 0$, so that $y = \frac{3}{5}$.

13. It suffices to consider the last three digits. There are $8 \times 7 \times 6 = 336$ possibilities, and we need to count how many of these are divisible by 8. To be divisible by 8, the unit digit must be 2, 4, 6 or 8. We consider these cases one by one.

- If the unit digit is 2, the tens digit must be 1, 3, 5 or 7 in order for the number to be divisible by 4. If the tens digit is 1 or 5, there are 3 choices of hundreds digit (e.g. if the tens digit is 1, then the hundreds digit must be 3, 5 or 7 as 312, 512 and 712 are divisible by 8); if the tens digit is 3 or 7, there are also 3 choices (namely, 4, 6, 8). Hence there are $3 \times 4 = 12$ possibilities in this case.
- If the unit digit is 4, the tens digit must be 2, 6 or 8, leading to 8 possibilities, namely, 624, 824, 264, 864, 184, 384, 584, 784.
- The case where the unit digit is 6 is essentially the same as the first case; 12 possibilities.
- The case where the unit digit is 8 leads to 10 possibilities (it is slightly different from the second case), namely, 128, 328, 528, 728, 168, 368, 568, 768, 248 and 648.

It follows that the answer is $\frac{12+8+12+10}{336} = \frac{1}{8}$.

14. Consider any 4 consecutive cards. There can be at most one '3' since there are at least three cards between any two '3's. On the other hand, there is at least one '3', for otherwise they have to be '1's and '2' subject to the required conditions, which can easily be seen to be impossible. In other words, there is exactly one '3' among any 4 consecutive cards.

Since $2011 \div 4 = 502.75$, the number of '3's must either be 502 or 503. Both bounds can be achieved (for the former case, consider 12131213…1213121; for the latter case, consider 31213121…3121312). Hence $m + M = 502 + 503 = 1005$.

15. Divide the students into groups according to their surname. If there are k students in a group, then every student in the group will write $k-1$ for the question on surnames. The same is true if we divide the students into groups according to birthdays.

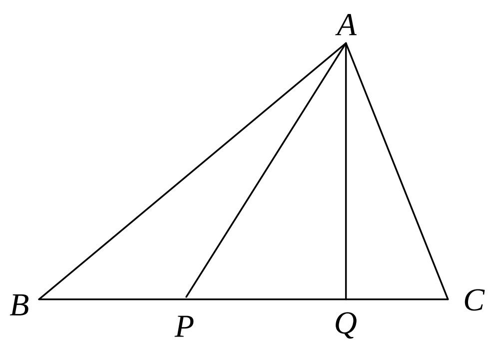
Since each of 0, 1, 2, …, 10 has appeared as an answer, there is at least one group of each of the sizes 1, 2, 3, …, 11. As each student belongs to two groups (one ‘surname group’ and one ‘birthday group’), the number of students must be at least $(1+2+3+\cdots+11)\div 2=33$.

Finally, it is possible that there are 33 students when, for instance, the answers to the ‘surname question’ are all 2, 8, 9, 10 (i.e. there are four ‘surname groups’ with 3, 9, 10, 11 students) and the answers to the ‘birthday question’ are all 0, 1, 3, 4, 5, 6, 7 (i.e. there are seven ‘birthday groups’ with 1, 2, 4, 5, 6, 7, 8 students). Hence the answer is 33.

16. Let m be the common root of $x^2+ax+1=0$ and $x^2+bx+c=0$. Clearly $m\neq 0$. Since $c\neq 1$, we must have $a\neq b$ as $m^2+am+1=m^2+bm+c=0$. This also gives $m=\frac{c-1}{a-b}$. Similarly, if n is the common root of $x^2+x+a=0$ and $x^2+cx+b=0$, then $n=\frac{a-b}{c-1}$. It follows that $mn=1$. As m is a root of the equation $x^2+ax+1=0$, which has product of roots 1, n is also a root of this equation.

Now n is a common root of the equations $x^2+ax+1=0$ and $x^2+x+a=0$, so we have $n^2+an+1=n^2+n+a=0$, which simplifies to $(a-1)(n-1)=0$. Clearly $a\neq 1$, for otherwise the equation $x^2+ax+1=0$ will have no real root. Thus we must have $n=1$ and $n^2+n+a=0$ gives $a=-2$. Since $mn=1$, we have $m=1$ and $m^2+bm+c=0$ gives $b+c=-1$. Thus $a+b+c=-3$.

17. By scaling we may assume $AC=1$ and $AB=2$. Then the cosine law asserts that $BC=\sqrt{5+4\cos 68^\circ}$. Since $CP=\frac{BC^2-AB^2}{BC}$ and $CQ=\frac{BC^2-3AC^2}{2BC}$, it can be shown that P is between B and Q. Since $AB^2=BC(BC-CP)=BC\times BP$, we have $\frac{AB}{BC}=\frac{PB}{BA}$. Together with the common $\angle B$, we have ΔABC ~ ΔPBA and so $\angle BAP$ = $\angle BCA$. It follows that



$$\angle APQ=\angle ABP+\angle BAP=\angle ABP+\angle BCA=180^\circ-112^\circ=68^\circ.$$

On the other hand, we have $\angle AQP=90^\circ$ since

$$AB^2-BQ^2=(2AC)^2-(BC-CQ)^2=4AC^2-BC^2+2BC\cdot CQ-CQ^2=AC^2-CQ^2.$$

Therefore $\angle PAQ=180^\circ-68^\circ-90^\circ=22^\circ$.

18. We consider two cases.

<u>Case 1: No green ball and blue ball are adjacent to each other</u>

In this case the red balls must either occupy the 1st, 3rd, 5th, …, 19th positions, or the 2nd, 4th, 6th, …, 20th positions. In each case there are $C_5^{10} = 252$ ways to arrange the green and blue balls. Hence there are altogether $252 \times 2 = 504$ ways in this case.

<u>Case 2: A green ball and a blue ball are adjacent to each other</u>

We place the 'green-blue pair' first. Since half of the total number of balls are red, and that no two red balls may be adjacent, the number of empty spaces before and after this pair must both be odd. Hence there are 9 choices of positions of this pair (2nd and 3rd, 4th and 5th, …., 18th and 19th). Of course there are 2 ways to arrange the two balls among the pair. After this pair is placed, the positions of the red balls are fixed (e.g. if the pair is placed in the 6th and 7th positions, then the red balls must be placed at the 1st, 3rd, 5th, 8th, 10th, …, 18th and 20th positions), and there are $C_4^8 = 70$ ways to arrange the remaining 4 green balls and 4 blue balls. Hence there are altogether $9 \times 2 \times 70 = 1260$ ways in this case.

Combining the two cases, the answer is $504 + 1260 = 1764$.

19. There are 1000000 possible 6-digit passwords, and we shall count how many of these have at least three identical consecutive digits. Let A (resp. B, C, D) denote the set of passwords in which digits 1 to 3 (resp. 2 to 4, 3 to 5, 4 to 6) are identical. Then we have

$$\begin{aligned}
&|A| = |B| = |C| = |D| = 10 \times 10^3 = 10000 \\
&|A \cap B| = |B \cap C| = |C \cap D| = 10 \times 10^2 = 1000 \\
&|A \cap C| = |B \cap D| = |A \cap D| = 10 \times 10 = 100 \\
&|A \cap B \cap C| = |B \cap C \cap D| = 10 \times 10 = 100 \\
&|A \cap B \cap D| = |A \cap C \cap D| = 10 \\
&|A \cap B \cap C \cap D| = 10
\end{aligned}$$

It follows that $\left|A \cup B \cup C \cup D\right| = 10000 \times 4 - (1000 \times 3 + 100 \times 3) + (100 \times 2 + 10 \times 2) - 10$
$= 36910$

and so the answer is $1000000 - 36910 = 963090$.

20. Dividing the first equation by the second, we get $\dfrac{x^3 - 5xy^2}{y^3 - 5x^2 y} = \dfrac{3}{4}$. Dividing both the numerator and denominator on the left hand side by y^3, and writing t for $\dfrac{x}{y}$, we have $\dfrac{t^3 - 5t}{1 - 5t^2} = \dfrac{3}{4}$, or

$$4t^3 + 15t^2 - 20t - 3 = 0 \qquad \text{……(*)}$$

Clearly, every solution (x_0, y_0) to the original system gives rise to a solution $t_0 = \frac{x_0}{y_0}$ to (*). Moreover, two different solutions (x_0, y_0) and (x_0', y_0') to the original system give rise to two different solutions $t_0 = \frac{x_0}{y_0}$ and $t_0' = \frac{x_0'}{y_0'}$ to (*). (This is because once t_0 is fixed, then so are x_0 and y_0; for instance the first equation together with $x_0 = t_0 y_0$ imply $y_0 = \sqrt[3]{\frac{21}{t_0^3 - 5t_0}}$.)

Writing $t_i = \frac{x_i}{y_i}$, we need to find the value of $(11-t_1)(11-t_2)(11-t_3)$, where each t_i is a different solution to (*), and hence $4t^3 + 15t^2 - 20t - 3 = 4(t-t_1)(t-t_2)(t-t_3)$. Consequently we have $(11-t_1)(11-t_2)(11-t_3) = \frac{4(11)^3 + 15(11)^2 - 20(11) - 3}{4} = 1729$.