International Mathematical Olympiad

Preliminary Selection Contest — Hong Kong 2006

Outline of Solutions

Answers:

1. 666	**2.** 1475	**3.** 60	**4.** $\dfrac{\sqrt{6}-\sqrt{2}}{2}$
5. $\dfrac{4\sqrt{3}}{3}$	**6.** $\sqrt{2}$	**7.** $\dfrac{27}{23}$	**8.** 32
9. 30000	**10.** 240	**11.** 4020	**12.** $\sqrt{6}$
13. 1505	**14.** 3456	**15.** 51	**16.** 30
17. $\dfrac{16\sqrt{17}}{17}$	**18.** 281	**19.** 2310	**20.** 13

Solutions:

1. Let the two numbers be x and y. Then we have $x+y=xy-2006$. Upon factorisation, we have

$$(x-1)(y-1)=2007.$$

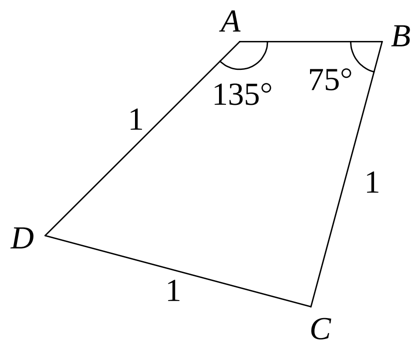
Since $2007=3^2\times 223$, it has 6 positive factors, namely, 1, 3, 9, 223, 669 and 2007. As one of x and y (say, x) is a perfect square, $x-1$ must be 1 less than a perfect square. Among 1, 3, 9, 223, 669 and 2007, only 3 has this property ($3=2^2-1$). Therefore we should take $x-1=3$ and $y-1=669$, i.e. $x=4$ and $y=670$. It follows that the answer is $670-4=666$.

2. Note that $\dfrac{2006n}{2006+n}=\dfrac{2006^2+2006n}{2006+n}-\dfrac{2006^2}{2006+n}=2006-\dfrac{2006^2}{2006+n}$.

Hence, in order for $2006n$ to be a multiple of $2006+n$, $2006+n$ must be a factor of 2006^2. As $1\le n\le 2005$, we have $2007\le 2006+n\le 4011$. As $2006^2=2^2\times 17^2\times 59^2$, the only factor of $2006+n$ between 2007 and 4011 is $59^2=3481$. Hence we must have $2006+n=3481$, or $n=1475$.

3. Let x be the normal walking speed of Peter and u be the speed of the escalator. Since speed is inversely proportional to time taken, we have $\frac{u+x}{u+2x}=\frac{30}{40}$, which gives $u=2x$. Let the answer be t seconds. Then we have $\frac{u}{u+x}=\frac{40}{t}$, so that $t=\frac{40(u+x)}{u}=\frac{40(2x+x)}{2x}=60$.

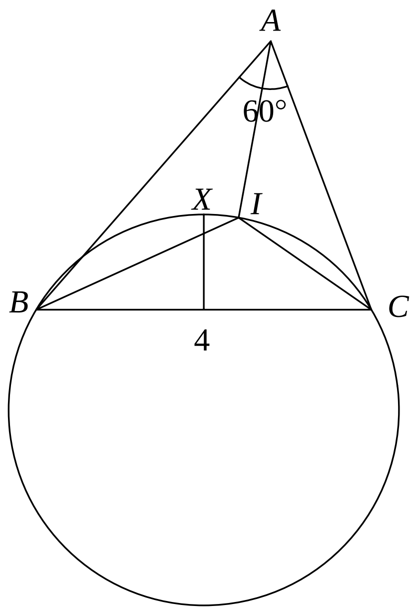
4. We claim that $AC=1$. If $AC>1$, then by considering ΔDAC we see that $\angle DAC$ must be smaller than 60° (recall that in a triangle, the side opposite a larger angle is longer), while $\angle CAB$ must be smaller than 75° by considering ΔABC. It leads to the contradiction $\angle DAB<135°$. In a similar way, we can get a contradiction when $AC<1$, thereby establishing the claim. Now ΔCAB is isosceles with $CA=CB=1$ and base angles 75°. Therefore



$$\begin{aligned} AB &= 2\cos 75° \\ &= 2\cos(45°+30°) \\ &= 2(\cos 45°\cos 30° - \sin 45°\sin 30°) \\ &= \frac{\sqrt{6}-\sqrt{2}}{2} \end{aligned}$$

5. We claim that X is the circumcentre of ΔABC. Assuming the claim, we know from the sine law that $\frac{4}{\sin 60°}=2AX$, so that $AX=\frac{4\sqrt{3}}{3}$.

It remains to prove the claim. Since X is on the perpendicular bisector of BC, we have $XB=XC$. It therefore suffices to show that $\angle BXC=2\angle BAC=120°$. But this is clear, because



$$\begin{aligned} \angle BXC &= \angle BIC \\ &= 180° - \angle IBC - \angle ICB \\ &= 180° - \frac{\angle ABC}{2} - \frac{\angle ACB}{2} \\ &= 180° - \frac{180° - \angle BAC}{2} \\ &= 120° \end{aligned}$$

6. Let $x=\cos\theta$ where $0\le\theta\le\pi$. Then $x+\sqrt{1-x^2}=\cos\theta+\sin\theta=\sqrt{2}\cos\left(\theta-\dfrac{\pi}{4}\right)$. Since the maximum value of the cosine function is 1, the maximum value of $x+\sqrt{1-x^2}$ is $\sqrt{2}$, and equality is attained when $\theta=\dfrac{\pi}{4}$, i.e. when $x=\dfrac{\sqrt{2}}{2}$.

Remark. A solution using calculus obviously exists.

7. Note that S_n and T_n, both being the sum of the first n terms of an arithmetic sequence, are both quadratic polynomials with constant term 0. Hence we may assume $S_n=An^2+Bn$ and $T_n=Cn^2+Dn$. Consequently, $a_n=S_n-S_{n-1}=2An+(B-A)$. Similarly $b_n=2Cn+(D-C)$. Plugging these into the given equation, we have

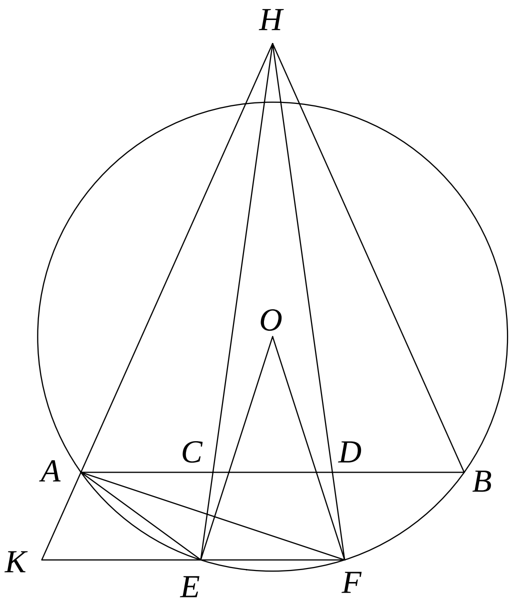
$$\frac{2An+(B-A)}{2Cn+(D-C)}=\frac{2n-1}{3n+1},$$

which gives $6An^2+(3B-A)n+(B-A)=4Cn^2+(2D-4C)n+(C-D)$. As this is true for all *n*, we equate the coefficients to solve for *A*, *B*, *C*, *D*. We get $B=0$, $C=\dfrac{3A}{2}$ and $D=\dfrac{5A}{2}$. Hence we have

$$\frac{S_9}{T_6}=\frac{A(9)^2+(0)(9)}{\left(\dfrac{3A}{2}\right)(6)^2+\left(\dfrac{5A}{2}\right)(6)}=\frac{27}{23}.$$

Remark. It is easy to guess the answer if one assumes $a_n=2n-1$ and $b_n=3n+1$.

8. Note that the point *G* is not useful except that it tells us that the chord *AB* subtends an angle of $48^\circ\times 2=96^\circ$ at the centre of the circle (which we call *O*) so that $\angle EOF=96^\circ\div 3=32^\circ$. We claim that *HA* // *OE* and *HB* // *OF*. Assuming the claim, then $\angle AHB=\angle EOF=32^\circ$ and hence the answer would be 32.



It remains to prove the claim. To show that *HA* // *OE*, we shall prove that *AF* is perpendicular to both *HA* and *OE*. That $AF\perp OE$ is clear, because *E* is the mid-point of arc *AF*. Now produce *HA* and *FE* to meet at *K*. First note that $AE=EF$, and $AC=CD$ implies $KE=EF$. In other words, the circle with diameter *KF* passes through *A*, so $AF\perp HA$. This shows *HA* // *OE*,

and by the same argument we have $HB \,//\, OF$,
thereby establishing the claim.

9. Suppose Tommy invests \$$a$, \$$b$ and \$$c$ on funds A, B and C respectively. Clearly, as long as a positive net profit can be guaranteed, we must take $a+b+c=90000$. Then we want to maximise

$$n=\min\{3a, 4b, 6c\}-90000$$

subject to the above conditions.

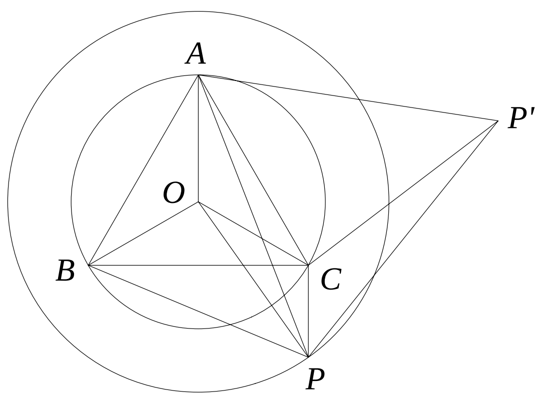
For maximality, we must have $3a=4b=6c$. This is because when $3a$, $4b$ and $6c$ are not all equal, say, if $3a<4b$ and $3a \le 6c$, then Tommy can shift some investments from fund B to fund A (or some to fund C as well, if necessary). In this way the value of $3a$ increases while the values of $4b$ and $6c$ will remain greater than or equal to $3a$ when the shift is sufficiently small. Therefore n increases, i.e. the value of n cannot be maximum when $3a$, $4b$ and $6c$ are not all equal.

Hence, in order to attain maximality, we want $a:b:c=\frac{1}{3}:\frac{1}{4}:\frac{1}{6}=4:3:2$. In this case, the maximum value of n is

$$3a-90000=3\times 90000\times\frac{4}{4+3+2}-90000=30000.$$

10. We first choose 4 boxes to place balls A, B, C, D. There are 10 possible combinations, namely, 1234, 1245, 1256, 1267, 2345, 2356, 2367, 3456, 3467 and 4567. Once the 4 boxes are chosen, there are 4 ways to place ball A, and then 1 way to place ball B (once ball A is placed there is only one possible position for ball B in order that balls C and D can be placed in boxes with consecutive numbers), and finally 2 ways to place balls C and D. There are 3 remaining boxes in which ball E can be placed. Hence the answer is $10\times 4\times 2\times 3=240$.

11. Let O be the common centre of the two circles, and let P' be a point such that $\Delta APP'$ is equilateral and that B and P' are on different sides of AP. Observe that $PA=PP'$ and $PB=CP'$ (as $\Delta BAP \cong \Delta CAP'$), so $\Delta PCP'$ is the desired triangle. Let $R=2007$, $r=2006$, $\angle POB = x$ and $\angle POC = y$. Then we have $x+y=120^\circ$ and $\frac{x-y}{2}=60^\circ-y$. Now



$$
\begin{aligned}
[PCP'] &= [APP'] - [APC] - [AP'C] \\
&= [APP'] - [APC] - [APB] \\
&= [APP'] - [ABPC] \\
&= [APP'] - [OAB] - [OAC] - [OBP] - [OCP] \\
&= \frac{\sqrt{3}}{4}\left[R^2 + r^2 - 2Rr\cos(120^\circ + y)\right] - \frac{\sqrt{3}}{4}(r^2 + r^2) - \frac{1}{2}Rr(\sin x + \sin y) \\
&= \frac{\sqrt{3}}{4}(R^2 - r^2) + \frac{\sqrt{3}}{2}Rr\cos(60^\circ - y) - \frac{1}{2}Rr \cdot 2\sin\frac{x+y}{2}\cos\frac{x-y}{2} \\
&= \frac{\sqrt{3}}{4}(R^2 - r^2) + \frac{\sqrt{3}}{2}Rr\cos(60^\circ - y) - \frac{\sqrt{3}}{2}Rr\cos(60^\circ - y) \\
&= \frac{\sqrt{3}}{4}(R^2 - r^2) \\
&= \frac{4013\sqrt{3}}{4}
\end{aligned}
$$

Hence $a+b+c = 4013+3+4 = 4020$.

Remark. It would be easy to guess the answer if one considers the special case where *A*, *O*, *P* are collinear.

12. Each face of the octahedron is an equilateral triangle of side length 6, the altitude of which is $6\sin 60^\circ = 3\sqrt{3}$. Consequently, half of the octahedron (i.e. the square pyramid in which every edge has length 6) has altitude $\sqrt{\left(3\sqrt{3}\right)^2 - 3^2} = 3\sqrt{2}$, i.e. the volume of the octahedron is

$$2 \times \frac{1}{3} \times 6^2 \times 3\sqrt{2} = 72\sqrt{2}.$$

On the other hand, if we let r be the radius of the inscribed sphere of the regular octahedron and A be the surface area of the octahedron, then the volume of the octahedron would also be equal to $\frac{1}{3}Ar$. Now $A = 8 \times \frac{1}{2} \times 6 \times 3\sqrt{3} = 72\sqrt{3}$. Hence we have $\frac{1}{3} \times 72\sqrt{3} \times r = 72\sqrt{2}$, which gives $r = \sqrt{6}$. This is also the greatest possible radius of a sphere which can be placed inside the tetrahedron.

13. Let $f(n) = \dfrac{n^2}{2006}$. For $n \le 1003$, we have

$$f(n) - f(n-1) = \frac{n^2}{2006} - \frac{(n-1)^2}{2006} = \frac{2n-1}{2006} < 1$$

Since $f(1) = 0$ and $f(1003) = \dfrac{1003^2}{2006} = 501.5$, the sequence contains every integer from 0 to 501 inclusive. On the other hand, when $n > 1003$,

$$f(n) - f(n-1) = \frac{2n-1}{2006} > 1.$$

Moreover, since

$$f(1004) = \frac{1004^2}{2006} = \frac{(1003+1)^2}{2006} = \frac{1003^2}{2006} + \frac{2\times 1003}{2006} + \frac{1}{2006} = 501.5 + 1 + \frac{1}{2006} > 502,$$

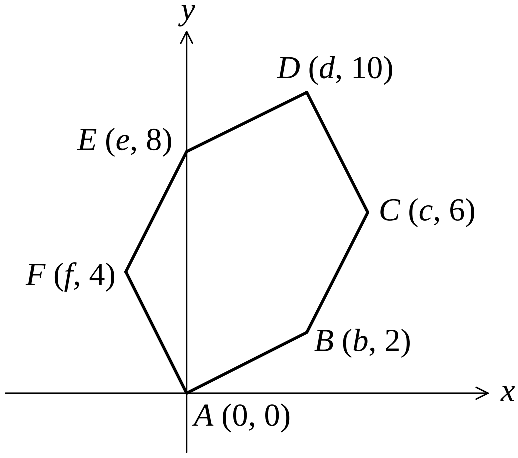
we see that $\left[\frac{1004^2}{2006}\right]$, $\left[\frac{1005^2}{2006}\right]$, …, $\left[\frac{2006^2}{2006}\right]$ are all distinct integers, all greater than 501. Therefore, the number of different integers in the sequence is $502 + 1003 = 1505$.

14. Such positive integers are all divisible by 9 since each has sum of digits $0+1+2+\cdots+9=45$. Hence we can change the number '11111' in the question to '99999' without affecting the answer. We denote such a 10-digit positive integer by $\overline{\text{ABCDEFGHIJ}}$. For this number to be divisible by 99999, a necessary and sufficient condition is that $\overline{\text{ABCDE}} + \overline{\text{FGHIJ}}$ should be divisible by 99999 (see Remark). Hence $\overline{\text{ABCDE}} + \overline{\text{FGHIJ}}$ must be exactly 99999. In other words, each of $\text{A}+\text{F}$, $\text{B}+\text{G}$, $\text{C}+\text{H}$, $\text{D}+\text{I}$ and $\text{E}+\text{J}$ must be equal to 9, i.e. we need only choose A, B, C, D, E. There are 9 choices for A as it cannot be zero (once A is chosen, F is also fixed). There are 8 choices for B as it cannot be equal to A and F (once B is chosen, G is also fixed). Similarly, there are 6, 4, 2 choices for C, D, E respectively, and at the same time H, I, J are fixed. Hence the answer is $9\times 8\times 6\times 4\times 2 = 3456$.

Remark. The divisibility test for 99999 is to put the digits into groups of 5 from the right, and then add up the numbers (as positive integers) in each group. The original number is divisible by 99999 if and only if the sum obtained is divisible by 99999. In the 10-digit case, this can be explained by the following congruence equation:

$$\begin{aligned}\overline{\text{ABCDEFGHIJ}} &= 100000\times\overline{\text{ABCDE}} + \overline{\text{FGHIJ}} \\ &\equiv 1\times\overline{\text{ABCDE}} + \overline{\text{FGHIJ}} \\ &= \overline{\text{ABCDE}} + \overline{\text{FGHIJ}} \qquad\qquad (\text{mod } 99999)\end{aligned}$$

15. Since *ABCDEF* is convex, it must be either *C* or *F* which has *y*-coordinate 4. Suppose *C* has *y*-coordinate 4. Then, since $AB = BC$, either *C* must be either (2*b*, 4) or (0, 4), both of which are not allowed (as the former implies that *A*, *B*, *C* are collinear while the latter implies that *ABCDEF* is concave). Hence *F* has *y*-coordinate 4. Furthermore, since *CD* // *AF*, the *y*-coordinate of *D* exceeds that of *C* by 4. It follows that the *y*-coordinates of *D* and *C* are 10



and 6 respectively, and thus the y-coordinate of E is 8.

Let $C = (c, 6)$, $D = (d, 10)$, $E = (e, 8)$ and $F = (f, 4)$. From the fact that $BC = CD$, we know that $b = d$, i.e. BD is vertical. As AB and ED are equal and parallel, $ABDE$ is a parallelogram, and hence AE is also vertical , so $e = 0$. Furthermore, ΔAEF and ΔBCD are congruent by the SSS condition. Note also that $f < 0$ by the convexity condition.

Now the area of $ABCDEF$ is equal to $2[AFE]+[ABDE] = 2\times\frac{1}{2}\times 8\times(-f)+8b = 8(b-f)$, so we must find the values of b and f. Let x be the side length of the hexagon. Computing the lengths of AB and AF respectively, we have

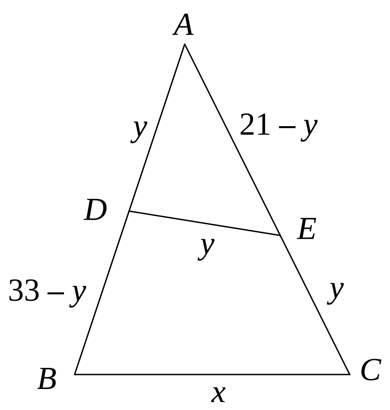
$$x^2 = b^2 + 4 = f^2 + 16 .$$

On the other hand, since $\angle FAB = 120°$, applying cosine law in ΔFAB gives

$$(b-f)^2 + (2-4)^2 = x^2 + x^2 - 2(x)(x)\cos 120° = 3x^2 .$$

Solving, we get $b = \frac{10}{\sqrt{3}}$ and $f = -\frac{8}{\sqrt{3}}$. Hence the area of $ABCDEF$ is $8(b-f) = 48\sqrt{3}$, and hence the answer is $48+3=51$.

16. Note that $y < 21$, and the triangle inequality in ΔADE implies $y + y > 21 - y$, or $y > 7$. It follows that y is between 8 and 20 inclusive.



Now, applying cosine law in ΔADE and ABC, we get

$$\cos A = \frac{y^2 + (21-y)^2 - y^2}{2y(21-y)}$$

and

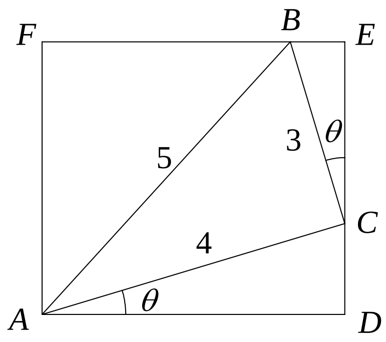
$$\cos A = \frac{33^2 + 21^2 - x^2}{2(33)(21)}$$

respectively. Equating and simplifying, we get

$$y(2223 - x^2) = 14553 = 3^3 \times 7^2 \times 11 .$$

Since $8 \le y \le 20$, the only possible values of y are 9 and 11. These correspond to $x^2 = 606$ and 900 respectively. As the former is not a perfect square, we take the latter which corresponds to $x = 30$.

17. Suppose that in ΔABC, $BC=3$, $AC=4$ and $AB=5$. Of course $\angle ACB$ = 90°. Without loss of generality, we fix A at the origin and assume that ΔABC lies in the first quadrant, and that the square to enclose ΔABC has sides parallel to the coordinate axes and has A as a vertex. Let $ADEF$ denote the smallest rectangle with sides parallel to the coordinate axes that can enclose ΔABC. Clearly, when $ADEF$ is a square, that is the smallest square that can enclose ΔABC.



Now suppose $ADEF$ is indeed a square with C on DE and B on EF as shown. Let $\angle CAD=\angle BCE=\theta$. We have $AD=4\cos\theta$ and $ED=3\cos\theta+4\sin\theta$. Equating, we get $\cos\theta=4\sin\theta$ and hence $\tan\theta=\frac{1}{4}$. Thus $AD=4\cos\theta=4\times\frac{4}{\sqrt{17}}=\frac{16\sqrt{17}}{17}$, which is the desired smallest possible side length.

18. We use (m, n) to denote the H.C.F. of m and n. Using the facts that $(m,n)=(m,n+km)$ for any integer k and $(m,n)=(2m,n)$ if n is odd, we have

$$\begin{aligned} g(n) &= (f(n),\ f(n+1)) = (70+n^2, 70+(n+1)^2) \\ &= (70+n^2,\ 2n+1) = (140+2n^2,\ 2n+1) \\ &= (140-n,\ 2n+1) = (280-2n,\ 2n+1) \\ &= (281,\ 2n+1) \\ &\le 281 \end{aligned}$$

Hence the largest possible value of $g(n)$ is 281. Finally, 281 is indeed attainable because $f(140)=70+140^2=281\times 70$ while $f(141)=70+141^2=281\times 71$. It follows that the answer is 281.

19. Let M be the L.C.M. of a_1, a_2, …, a_{11}, and set $m_i=\frac{M}{a_i}$. Then each m_i is a positive integer and $m_1>m_2>\cdots>m_{11}$. Furthermore, m_1, m_2, …, m_{11} form an arithmetic sequence and M is also the L.C.M. of m_1, m_2, …, m_{11}. Write $b_{11}=b$ and $b_i-b_{i+1}=d$, where b, d are positive integers. Note that b and d have to be relatively prime by the definition of M. We want to minimise

$$a_1=\frac{\operatorname{lcm}(b, b+d, \ldots, b+10d)}{b+10d}.$$

When $b=2$ and $d=1$, we find that $a_1=2310$. We prove below that a_1 cannot be any smaller, so that the answer would be 2310. Indeed, if $d\ge 2$, then

$$a_1 \ge \frac{\operatorname{lcm}(b+5d, b+6d, \dots, b+10d)}{b+10d} = \frac{(b+5d)(b+6d)\cdots(b+10d)}{2\cdot 2\cdot 2\cdot 3\cdot (b+10d)} \ge \frac{5\cdot 6\cdot 7\cdot 8\cdot 9}{2\cdot 2\cdot 2\cdot 2\cdot 3} d^4 > 2310 .$$

Hence we may assume $d = 1$. In this case

$$a_1 \ge \frac{\operatorname{lcm}(b+5, b+6, \dots, b+10)}{b+10} = \frac{(b+5)(b+6)\cdots(b+10)}{2\cdot 2\cdot 2\cdot 2\cdot 3\cdot (b+10)} \ge \frac{(b+5)(b+6)\cdots(b+9)}{48}$$

If $b \ge 4$, we see that $a_1 > 2310$. So it remains to check the cases b = 1, 2, 3. Among these, we find that $b = 2$ gives the minimum value $a_1 = 2310$.

20. Let $f(n)$ be the number of the card received by child *n*. Consider the sequence

$$1,\ f(1),\ f(f(1)),\ f(f(f(1))), \dots$$

Note that the term 2006 must eventually occur and the sequence can no longer be continued as $f(2006)$ is undefined. Furthermore, if we let *S* be the set of integers from 1 to 2006 which have not occurred in the above sequence, then we see that $f(S) = S$. It easily follows that every child in *S* must receive the card with the same number as his own.

It therefore remains to count the number of such sequences, starting from 1 and ending in 2006, and each term (except for the first one) is a multiple of the previous term. Note that $2006 = 2\times 17\times 59$, and every positive factor of 2006 is of the form $2^a 17^b 59^c$ and can be denoted by the three-tuple (a,b,c) where each of *a*, *b*, *c* is 0 or 1. The problem is thus reduced to counting the number of sequences from (0,0,0) to (1,1,1) for which each coordinate must be greater than or equal to the corresponding coordinate in the previous term.

If we increase the three coordinates one by one, we get $3! = 6$ such sequences. If we first increase one of the coordinates and then increase the other two at once, we get $C_1^3 = 3$ such sequences. If we first increase two of the coordinates at once and then increase the remaining one, we get $C_2^3 = 3$ such sequences. Finally, we get 1 such sequence if we jump from (0,0,0) to (1,1,1) directly. It follows that the answer is $6+3+3+1=13$.