

USING PROBLEM-POSING AS AN ASSESSMENT TOOL

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ABSTRACT

Students who are mathematically promising need to go beyond problem solving to problem posing and finally to creating mathematical problems. It is only at this highest level of creation that students will begin to realize their true potential and experience the excitement of mathematical discovery and research. Considering the pedagogical benefits of problem posing, should the tasks not also be embedded into the classroom assessment? This ongoing action research explored the use of problem-posing tasks as an assessment tool to examine students' thinking processes, understandings and competencies. Based on the analytic scheme developed by Silver and Cai (1996), this paper describes how the team of teacher researchers implemented the problem-posing tasks and their analysis of the problems posed by 120 high ability students from a secondary school. The students' performances were analyzed and evaluated in the light of problem complexity. Initial results from the study have shown that the use of problem-posing tasks not only provided opportunities for students to demonstrate what they know and can do with their mathematical knowledge; it also allowed the teacher to observe patterns in students' mathematical learning and thinking.

INTRODUCTION

High ability mathematical students need to be nurtured and their talents developed through rigorous curriculum, acceleration and breadth of conceptual mathematical understandings. Sheffield (2003) suggested that students be directed along a mathematical continuum from novice to expert as follows: innumerates, doers, computers, consumers, problem solvers,

problem posers and creators. The essential skills required of a learner to move from a lower level to a higher level in this continuum are clearly featured in the Singapore's Mathematics Curriculum Framework (Figure 1), from the Ministry of Education, Singapore. According to the framework, the teacher would place problem solving at the centre of their instructional goal.

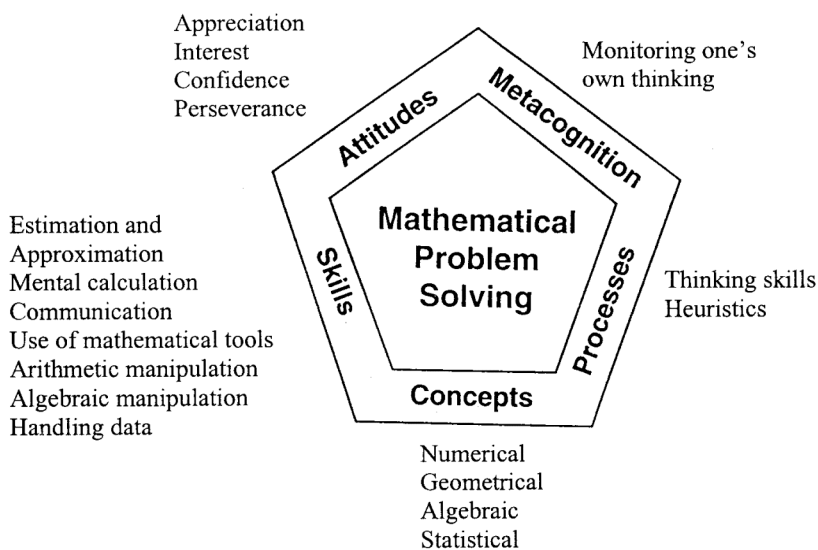


Figure 1

Many problem-solving activities aimed to help students acquire mathematical skills and master important concepts. Problem-posing, seen as an important part of the problem-solving task, takes a complementary role in further portraying mathematics as an empirical subject – integrated and creatable. With purposeful and consistent planning, opportunities for students to reason, reflect on their own thinking and make connections between mathematics and the real world often spring from such creative experiences. More important for the purpose of this study, problem posing activities potentially reveal much about the students' understandings, knowledge, skills and attitudes as they interact with the situation presented to them.

PROBLEM POSING

Problem posing is an important companion to problem-solving. It refers both to the generation of new problems from a mathematical context and to the reformulation of solution to a given problem (Silver, 1994). In the first instance, the problem poser would usually need to consider the nature of the context and possible solution paths to the problem posed. This process of problem generation and consideration for multiple solution paths provides excellent opportunities for fostering divergent and flexible thinking. Such thinking habits not only enhanced problem-solving skills, but also help to reinforce and enrich basic mathematics concepts. Hence from a pedagogical perspective, problem posing activities potentially present themselves as powerful assessment tools (English, 1997; Lowrie, 1999).

ASSESSMENT WITH PROBLEM POSING

Good assessment is about expanding the assessment repertoire because no single form is sufficient. Since every assessment tool faces different issues related to reliability and validity, each of them has its strengths, weaknesses and its place in educative assessment. More importantly, good assessment practice is about applying the assessment repertoire to provide students with multiple opportunities, in varying contexts, to demonstrate what they know, understand and can do in relation to the stipulated learning outcomes.

Despite the recent development in incorporating problem posing as an instructional tool, far less attention has been paid to problem posing and assessment. Cai and Silver (2005) suggested the possibility of expanding the role of problem posing as a classroom assessment to evaluate

students' understanding and proficiency. It is the use of problem posing activities to inform teachers about students' learning that forms the impetus for this action research on exploring the use of problem posing as a formative assessment tool.

In order to effectively use problem posing in the classroom as a generative activity from which information about the students' engagement, competencies and areas of improvement can be drawn, it is essential for teachers to understand its value as a formative assessment. There should be a common understanding of the main goal of formative assessment. These assessments, whichever form that they take, aim to help both teachers and learners find out how much learning has taken place, and then use the results to chart the process of the learners' learning. Teachers can then point directions to where their learners' strengths can be further developed and at the same time identify those areas of improvement. It is this interest in the process of gathering data about students' growth in learning mathematics for improving classroom instruction and learning that drives this study.

THEORETICAL PERSPECTIVES

There are several ways to evaluate students' responses to problem posing tasks. Silver and Cai (2005) noted that

“Because of the open-ended nature of such tasks, there is often considerable variability in the responses that students generate. Although this aspect is desirable from an instructional perspective, it can often present challenges from an assessment perspective.” (p. 131)

Based on the principles of assessment, teachers must make decisions about assessment tasks by first considering their instructional goals and the form of the tasks that can potentially provide evidence of attaining these goals. The particular features of the tasks related to the instructional goals contribute largely to the set of criteria teachers use for evaluating students' performance in the assessment. In their study on assessing students' mathematical problem posing, Silver and Cai (2005) identified three criteria that are commonly applicable to most problem posing tasks: *quantity*, *originality* and *complexity*.

Quantity refers to the number of correct responses generated from the problem posing task. Counting the number of (correct) responses may be deemed by many as a trivial way of evaluating students' responses to generative activities such as problem posing. Nevertheless, the fluent generation of responses can potentially inform the teacher about students' characteristics such as creativity. Originality is also another feature of responses that can possibly be used as a criterion to measure students' creativity.

Problem *complexity* can be examined from various perspectives. Silver and Cai (2005) identified four facets of problem complexity – sophistication of the mathematical relationships embedded in problems, problem difficulty, linguistic complexity and mathematical complexity. Of particular interest to this study is mathematical complexity. Mathematical complexity refers to the cognitive demands of the task. It can be categorized as low, moderate, or high. Each level of complexity includes aspects of knowing and doing mathematics, such as reasoning, performing procedures, understanding concepts, or solving problems. The levels of complexity form an ordered description of the demands a problem may make on a student. Problems categorized at

the low level of complexity, may require a student to recall a property. At the moderate level, a problem may require the student to make a connection between two properties; at the high level, a problem may require a student to analyze the assumptions made in a mathematical model.

OBJECTIVE OF THE STUDY

As part of the research study on using problem posing as a formative assessment for learning, this ongoing action research first explore the range of different problems posed by students. Samples of their work will be analyzed for studying the types of problems posed, and identifying patterns in their mathematical learning and thinking.

METHODS USED IN THE STUDY

Participants

Thirty-two secondary one pupils and fifty secondary three girls participated in this study. From their consistently outstanding performance and interest in Mathematics, these students are identified to be mathematically promising and are placed in advanced classes. Despite their previous experiences with problem solving, none of the participants have prior exposure to problem posing activities.

Procedure

The preparation for using problem posing tasks as formative assessment began with the teacher modeling the problem posing behaviour during instruction. Some of the problem-generating strategies were presented to the students during the course on Arithmetic for the secondary one students, and Algebra for the secondary three students. The techniques of changing a problem to

create new ones included: changing the numbers and changing the operations, and removing or adding conditions in an existing problem. This exposure to problem posing techniques took place over a period of about six weeks.

At the end of the course, the students worked collaboratively with each other to pose problems to a task involving semi-structured situations. The task was open-ended in the sense that the students could pose any problems they wished so long as the problems met the constraints delineated in the task. While the secondary one students were assessed on the ability to apply their pre-knowledge of proportions and percentages in constructing word problems set within realistic and meaningful context; the secondary three students' knowledge of inequalities was evaluated based on their ability to add conditions in an existing problem to create a new optimization problem.

For this group of high ability students who have no prior experience in problem posing, teacher facilitation is an important part of the process of assessment for learning. During the course of work, the different levels of mathematical complexity in the problems were constantly referred to and analyzed based on the given rubric.

For the purpose of evaluating the students' mathematical understanding and cognitive processes, the students' responses to the problem posing task were assessed based mainly on an important characteristics of the problems posed - mathematical complexity. Figure 2, adapted from *Mathematics Framework for the 2005 National Assessment of Educational Progress (2005)*, describes the three different levels of mathematical complexity. It served as a guide for students

to pose quality problem with high mathematical complexity. It also provides students with a possible course of actions in order to move from a category to the next higher level.

Low complexity	Moderate complexity	High complexity
<p>This category relies heavily on the recall and recognition of previously-learned concepts. Items typically specify what the solver is to do, which is often to carry out some procedure that can be performed mechanically. It leaves little room for creative solutions. The following are some, but not all, of the demands that items in the low-complexity category might make:</p>	<p>Items in the moderate-complexity category involve more flexibility of thinking and choice among alternatives than do those in the low-complexity category. They require responses that may go beyond the conventional approach, or require multiple steps. The solver is expected to decide what to do, using informal methods of reasoning and problem-solving strategies. The following illustrate some of the demands that items of moderate complexity might make:</p>	<p>High-complexity items make heavy demands on solver, who must engage in more abstract reasoning, planning, analysis, judgment, and creative thought. A satisfactory response to the item requires that the solver think in an abstract and sophisticated way. The following illustrate some of the demands that items of high complexity might make:</p>
<ul style="list-style-type: none"> • Recall or recognize a fact, term, or property • Compute a sum, difference, product, or quotient • Perform a specified procedure • Solve a one-step word problem • Retrieve information from a graph, table, or figure 	<ul style="list-style-type: none"> • Represent a situation mathematically in more than one way • Provide a justification for steps in a solution process • Interpret a visual representation • Solve a multiple-step problem • Extend a pattern • Retrieve information from a graph, table, or figure and use it to solve a problem 	<ul style="list-style-type: none"> • Describe how different representations can be used to solve the problem • Perform a procedure having multiple steps and multiple decision points • Generalize a pattern • Solve a problem in more than one way • Explain and justify a solution to a problem

	<ul style="list-style-type: none"> • Interpret a simple argument 	<ul style="list-style-type: none"> • Describe, compare, and contrast solution methods • Analyze the assumptions made in solution • Provide a mathematical justification
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Figure 2

Instrument

The format of the problem posing task administered in the study is similar to those developed by Silver and Cai (2005). The stimulus situation involved some quantitative information which the students will creatively use to craft problems related to inequalities. The task requirements for the secondary one and secondary three students are shown in Figure 3 and Figure 4 respectively.

Task objective:

Pose mathematical problems to demonstrate your competency in constructing authentic word problems involving one or a combination of the following aspects: percentages, hire purchase, simple and compound interest, money exchange and taxation. Solve the problem you have posed, reformulating it where necessary.

Figure 3

Task objective:

From the information below, construct mathematical problems, and solve them, to demonstrate your competency in using the basic rules for manipulating inequalities to simplify and solve simultaneous inequalities or inequalities involving linear, quadratic or modulus functions.

A gardener is planting a new orchard. The young trees are arranged in the rectangular plot, which has its longer side measuring 100m.

Figure 4

The students were given the opportunity to assess their own problems, and then evaluate other problems posed by their peers. This process of self- and peer assessment served to help them internalize the characteristics of quality work

DISCUSSIONS OF FINDINGS

Classification of problems

The following classification of problems posed by students is based on the analytic scheme proposed by Silver and Cai (1996), as shown in Figure 5, to examine the problem posing of middle school students.

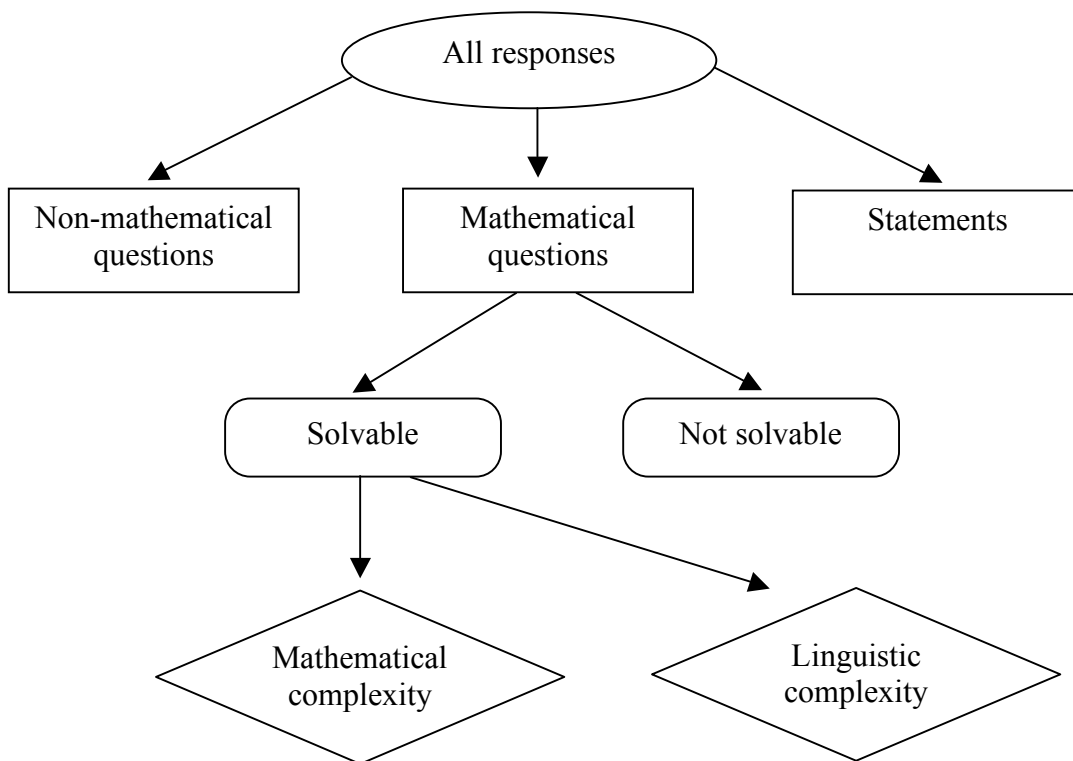


Figure 5

According to the schema, the problems that are statements and non-mathematical questions are first sieved out, before focusing on those that are mathematical problems. Within the set of mathematical problems posed by students, problems that are solvable are identified for further examination based on the nature and mathematical complexity of the problems posed.

In the collation of the problems posed by the students, this first stage of data collection saw all the students successfully constructing mathematical questions. Figure 6 shows the proportion of the problems that are solvable, their varying level of mathematical complexity and additional information about the extent to which the problem posed are interesting and challenging.

	Secondary 3	Secondary 1
Mathematical questions	100%	100%
Solvable	78%	81%
Mathematical complexity level 1	67%	81%
Mathematical complexity level 2	30%	13%
Mathematical complexity level 3	3%	6 %
Interesting	58%	81%
Challenging	50%	19%

Figure 6

In classifying the problems posed by students, it was observed that the high number of unsolvable problems was due to unclear wording in the problem, important assumptions were not stated and the use of overly complex algebraic expressions. Of those problems that are solvable, more than half of them showed low mathematical complexity. Although many of these problems

appeared to be somewhat complex, the demand on the mathematical skills used to solve the problem is fairly low. Sometimes, the problems can be more easily solved using simple arithmetic. Moreover, many of the problems posed in this category were reflective of the routine questions commonly found in textbooks. Some of such responses are shown in Figure 7.

Sample 1 of problems showing low mathematical complexity
Mrs Jill wants to buy a car for \$120 000. She is offered loans by Bank A and Bank B. Bank A allows her to pay in 20 monthly installments, with 15% p.a. simple interest. For Bank B, Mrs. Jill has to pay \$12 000 as deposit, and the rest in 16 monthly installments, at an interest rate of 10% p.a., compounded annually. Which bank offers a better deal? How much does she save?
If a fence around the orchard measures more than 330m, and the area of the orchard is not more than 7000m^2 , find the range of values of the shorter side.

Figure 7

For problems that were classified under moderate mathematical complexity, they generally reflect real-world contexts, and most students find them more interesting to begin with. The teacher's observational notes also indicated that the groups working on those problems with real-world context not only did not confine their discussions to mathematics *per se*, but also concerned themselves with current information and realistic views about the world. These groups were observed to be more engaged in their participation in the problem posing activity and they also took a longer time to complete their tasks. Some of such responses are shown in Figure 8.

Sample 2 of problems showing moderate mathematical complexity

Ms X had S\$83264, which she wanted to change to Z\$ to buy an item in Zork. The exchange rate is at Z\$1 = S\$4.1632 (selling rate) and Z\$1 = S\$4.0071 (buying rate). She bought the item using hire purchase, paying a 20% deposit and was charged a 3.2% simple interest rate per annum over a period of 36 months. She had S\$36.993.55 leftover when she converted the money she had left after paying the total balance. What was the selling price of the item in Z\$?

The gardener decides to divide the plot of land into three sections for growing three different types of plants. It is given that section C is representative of a quadrant and the area of section C is bigger than that of section A. The various sections require different types of soils of different prices. If the gardener has a budget of \$400 for buying soil for the orchard, what is the maximum value of x ?

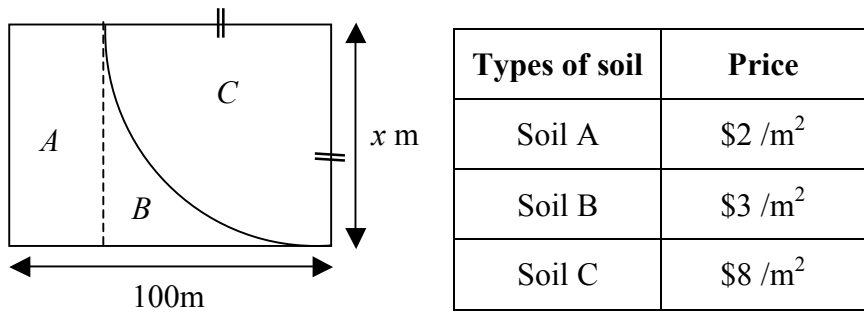


Figure 8

Problems posed by students have varying cognitive demands. Those that are classified under high mathematical complexity are especially appealing to students who demonstrate sound conceptual knowledge and a wide repertoire of mathematical skills. For students to craft problems that require abstract reasoning, judgment and analysis, it is not unreasonable to expect them to demonstrate these traits. The teacher also observed that the students who pose problems

of this quality were observed to have a relatively positive disposition towards problem solving. Some of such responses are shown in Figure 9.

Sample 3 of problems showing high mathematical complexity
It is given that the width of the orchard is 50m. Starting from point B , a worker P walked along the edge in a clockwise direction and back to B at a speed of 2m/s. Another worker, Q , started from point A and walked along the edge in the clockwise direction and back to point A at a speed of 1 m/s. What is largest possible area of triangle BPQ ?

Figure 9

In general, most of the high ability students were able to generate problems that exceed the degree of difficulty and novelty of those problems found in the textbooks.

Patterns of mathematical learning and thinking

The students' attempts to construct a well-defined, solvable problem within an appropriate context with suitable objects and event and yet embodies specific concepts proved to be a cognitively demanding experience. There are some students who found it a challenge to apply the specific concepts across a wide range of problem contexts. This difficulty could be due to a number of factors, such as having a narrow perspective of a concept or perceiving the concept as an isolated idea. For example, many students mistakenly thought that having a "maximum" or "minimum" problem would constitute a problem involving inequalities. Another possible reason could be their limited exposure to different contexts in which a mathematical concept or a combination of concepts can operate. There were also students who showed appreciation for the

opportunity to make connections between different mathematical ideas within a context, and are able to weave them together to construct a rich task.

IMPLICATIONS FOR FURTHER RESEARCH

From this ongoing action research, much information has been gathered about how teachers can systematically plan for problem posing activities so as to directly examine the students' understandings of mathematical concepts in the problem posed and competence in problem solving. This informal way of gathering information about students' thinking processes, strategies and their developing mathematical understandings is present in most formative assessments. In addition, it presented the students with the full array of tasks that require them to be effective performers with acquired knowledge. From the affective point of view, the students' beliefs and other attributes such as their desire to take risk and being open to constructive feedback were revealed. Hence, problem posing can potentially be used as an assessment tool for effective teaching and learning of mathematics.

CONCLUSION

This study on the types of problems posed by students and their thinking processes in problem posing served as a good starting point to explore and build a taxonomy of thinking processes, for secondary school students, that are related to generative activities such as problem posing. This can potentially give teachers a more complete account of the students' mathematical competencies and understandings. With its potential to foster more diverse and flexible thinking, enhance problem solving skills, broaden perceptions of mathematics and enrich and consolidate

basic concepts (Brown & Walter, 1993; English, 1996; Silver & Burkett, 1993; Simon, 1993), it is highly viable for problem posing to be used as an assessment tool for learning mathematics.

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