International Mathematical Olympiad

Preliminary Selection Contest 2010 — Hong Kong

Outline of Solutions

Answers:

1. 1000	**2.** 311550	**3.** 3888	**4.** 88
5. 501	**6.** 721	**7.** 2024	**8.** 32768
9. –3100	**10.** $3\sqrt{3}-\pi$	**11.** $2+\sqrt{3}$	**12.** $2\pi+3-3\sqrt{3}$
13. 203	**14.** 10	**15.** 213	**16.** 132
17. $8\sqrt{3}$	**18.** $\frac{329}{100}$	**19.** $\frac{\sqrt{15}}{9}$	**20.** –5

Solutions:

1. Note that $f(n)=n^3-(n-1)^3$. It follows that

$$f(1)+f(2)+\cdots+f(2010)=(1^3-0^3)+(2^3-1^3)+\cdots+(2010^3-2009^3)=2010^3=201^3\times 1000.$$

Since 201^3 has unit digit 1, it is clear that the answer is 1000.

Remark. Even without realising $f(n)=n^3-(n-1)^3$, one could still easily observe that $f(1)+f(2)+\cdots+f(n)=n^3$ by computing this sum for some small n, which is a natural approach given that the question asks for something related to $f(1)+f(2)+\cdots+f(2010)$.

2. The digit which is even must be the unit digit 0. Hence it remains to find the smallest integer positive k for which all digits of $201k$ are odd.

Clearly we must have $k\geq 150$, for if $k<100$ then the hundreds digit of $201k$ is even (same as the unit digit of $2k$); if $100\leq k<150$ the ten thousands digit of $201k$ is 2 which is even. Furthermore k must be odd as the unit digit of $201k$ has to be odd. Hence we try $201\times 151=30351$, $201\times 153=30753$ and $201\times 155=31155$ to get the answer is 311550.

3. We consider the following cases.

- If n is of the form 2^a, then we have $n > 2\times 1200 = 2400$, and the smallest n of this form is $2^{12} = 4096$.
- If n is of the form 3^b, then we have $n > 3\times 1200 = 3600$, and the smallest n of this form is $3^8 = 6561$.
- If n is of the form $2^a 3^b$, then we have $n > 3\times 1200 = 3600$, and after some trial (for example, by first fixing a) we find that the smallest n of this form is $2^4 \times 3^5 = 3888$.
- If n contains a prime factor greater than or equal to 5, then we have $n > 5\times 1200 = 6000$.

Combining the above cases, the answer is 3888.

4. First, note that $N > 87$. This is because if there are 75 green, 12 red, 12 white and 12 blue balls in the bag, we do not necessarily get balls of at least three different colours when at most 87 balls are drawn (for instance we may only get green and red balls), and this combination of balls satisfies the condition 'if 100 balls are drawn, we can ensure getting balls of all four colours' because, when 100 balls are drawn, only 11 balls are missing and hence no colour can be missing from the balls drawn as there are at least 12 balls of each colour.

Next we show that $N = 88$ is enough. The given condition ensures at least 12 balls of each colour (for otherwise if there are, say, 11 or fewer red balls, it is possible for all of them to be missing when 100 out of the 111 balls are drawn, contradicting the given condition), so when 88 balls are drawn, 23 are left over and it is impossible to have 2 colours missing (which requires at least 24 balls to be left over). It follows that the answer is 88.

5. We have $a-b \geq 1$ and $c-d \geq 1$. As

$$2010 = a^2 - b^2 + c^2 - d^2 = (a+b)(a-b) + (c+d)(c-d) \geq (a+b) + (c+d) = 2010,$$

equality holds throughout and hence we must have $a-b = c-d = 1$. This gives $(b+1)+b+(d+1)+d = 2010$, or $b+d = 1004$, where b and d are positive integers with $b \geq d+2$. Hence we get 501 choices for (b, d), namely, (1003, 1), (1002, 2), …, (503, 501), and hence 501 choices for (a, b, c, d), namely, (1004, 1003, 2, 1), (1003, 1002, 3, 2), …, (504, 503, 502, 501).

6. Note that $(x-1)^2 + 1 \leq P(x) \leq 2(x-1)^2 + 1$ for any real number x. Hence we must have $P(x) = a(x-1)^2 + 1$ for some $1 \leq a \leq 2$. Since $P(11) = 181$, we have $181 = a(11-1)^2 + 1$, or $a = \frac{9}{5}$. It follows that $P(21) = \left(\frac{9}{5}\right)(21-1)^2 + 1 = 721$.

7. Let n be the integer. Note that there must be no integer k for which $\frac{n}{144} \le k \le \frac{n}{135}$, for if such integer exists, we must be able to write n as the sum of k numbers, each between 135 and 144. For instance, if $n = 2500$, then $\frac{n}{144} \approx 17.4$ and $\frac{n}{135} \approx 18.5$. We may then take $k = 18$. Since $135 \times 18 = 2430$ and $144 \times 18 = 2592$, to form 2500 we may start by using 135 only (and we are short by 70), and then change 14 of the 135 to 140, thus getting $2500 = 135 \times 4 + 140 \times 14$.

   If $\frac{n}{135} - \frac{n}{144} \ge 1$, then the above-mentioned integer k must exist. Hence $\frac{n}{135} - \frac{n}{144} < 1$ and so $9n < 135 \times 144$, or $n < 135 \times 16 = 2160$. When n is slightly less than 2160, we can still take $k = 15$ until n is less than $135 \times 15 = 2025$.

   Indeed, the answer is 2024. This is because with 14 summands, the value of n is at most $144 \times 14 = 2016$; with 15 summands, the value of n is at least $135 \times 15 = 2025$. It follows that 2024 cannot be expressed in the form described in the question.

8. Let $n = 1000a + b$, where $0 \le b \le 999$. By removing the last three digits of n, we get a. Hence we have $a^3 = 1000a + b$, or $b = a(a^2 - 1000)$. Since $b \ge 0$, we have $a^2 \ge 1000$ and hence $a \ge 32$. If $a \ge 33$, then $b \ge 33(33^2 - 1000) > 1000$ which is impossible. Hence we must have $a = 32$, which gives $b = 32(32^2 - 1000) = 768$ and hence $n = 32768$.

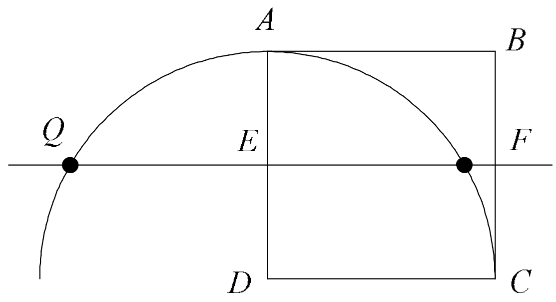
9. Let α and β be the roots of the equation $10x^2 + px + q = 0$. Then we have $\alpha + \beta = -\frac{p}{10}$ and $\alpha\beta = \frac{q}{10}$, and so $2010 = p + q = (-10)(\alpha + \beta) + 10\alpha\beta$, which gives $\alpha\beta - \alpha - \beta = 201$ or $(\alpha - 1)(\beta - 1) = 202$. As α and β are positive integers, the two factors $\alpha - 1$ and $\beta - 1$ may be 1 and 202, or 2 and 101. Hence α and β may be 2 and 203, or 3 and 102. These correspond to $p = -2050$ and $p = -1050$ respectively. It follows that the answer is –3100.

10. Note that the hexagon must have as its vertices the midpoints of six edges of the cube. Hence the side length of the hexagon is $\sqrt{2}$ and its area is $6 \cdot \frac{1}{2} \cdot (\sqrt{2})^2 \cdot \sin 60° = 3\sqrt{3}$.

    On the other hand, P passes through the centre of the inscribed sphere by symmetry, so it cuts out a cross section of radius 1 (which is the radius of the inscribed sphere), whose area (which is contained entirely inside the hexagon) is π.

    Therefore the answer is $3\sqrt{3} - \pi$.

11. As Q is the circumcentre of ΔBPC, it lies on the perpendicular bisector of BC, and we have $PQ = CQ$. As D is the circumcentre of ΔPQA, we have $DQ = DA = 1$ and hence Q lies on the circle with centre D and radius 1. These two loci intersect at two points, giving two possible positions of Q. We choose the one outside ΔABC in order to maximise PQ (= CQ).



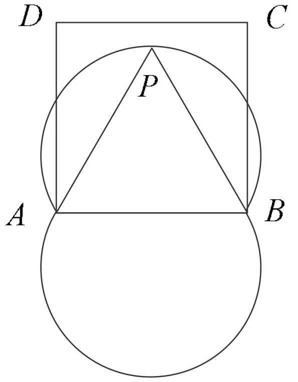
Let E and F be the midpoints of AD and BC respectively. Since ΔAPQ is equilateral with side length 1, we have $EQ = \frac{\sqrt{3}}{2}$. Appling Pythagoras Theorem in ΔCFQ, we have

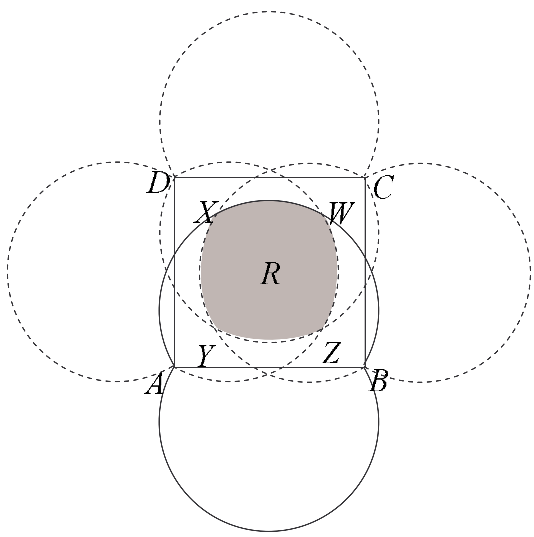
$$PQ^2 = CQ^2 = CF^2 + FQ^2 = \left(\frac{1}{2}\right)^2 + \left(1 + \frac{\sqrt{3}}{2}\right)^2 = 2 + \sqrt{3}.$$

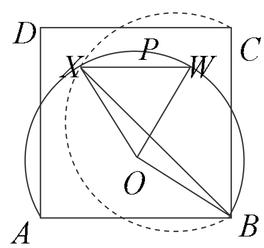
12. As shown in the left hand figure below, the set of points P for which $\angle APB \geq 60°$ is the union of two segments (the major segment formed by AB in the circumcircle of ΔABP, where P may take one of the two positions for which ΔABP is equilateral). Hence the red region R is the intersection of the four such congruent shapes (denoted by $XYWZ$ in the centre figure below).

Let O be the centre of major arc $AXWB$. By symmetry, $\angle XBA$ = 45°, and so $\angle XOA$ = 90°. Similarly, $\angle WOB$ = 90°. Also, $\angle AOB = 120°$ as angle at centre is twice angle at circumference. Thus $\angle XOW$ = 360° – 90° – 120° – 90° = 60°.

As shown in the right hand figure below, R consists of four identical minor segments (like XPW) and a square $XYZW$. Since $AB = 3$, each segment has radius $\sqrt{3}$. Thus the square $XYZW$ has area 3 and each segment has area $\frac{1}{6}\pi\left(\sqrt{3}\right)^2 - \frac{\sqrt{3}}{4}\left(\sqrt{3}\right)^2 = \frac{\pi}{2} - \frac{3\sqrt{3}}{4}$. It follows that the answer is $3 + 4\left(\frac{\pi}{2} - \frac{3\sqrt{3}}{4}\right) = 2\pi + 3 - 3\sqrt{3}$.







13. We only have to consider which points are to be connected by red lines, and the rest of the lines must be blue. Note that the restriction means that the 6 points must be divided into some number of groups, such that two points are joined by a red line if and only if they are in the same group. So the question becomes counting the number of such groupings, and we count using the pattern of grouping, as follows:

Pattern	Number
6	1
5-1	$C_1^6 = 6$
4-2	$C_4^6 = 15$
4-1-1	$C_4^6 = 15$
3-3	$C_3^6 \div 2 = 10$
3-2-1	$C_3^6 \times C_2^3 = 60$
3-1-1-1	$C_3^6 = 20$
2-2-2	$C_2^6 \times C_2^4 \div 3! = 15$
2-2-1-1	$C_2^6 \times C_2^4 \div 2 = 45$
2-1-1-1-1	$C_2^6 = 15$
1-1-1-1-1-1	1

Hence the answer is $1+6+15+15+10+60+20+15+45+15+1=203$.

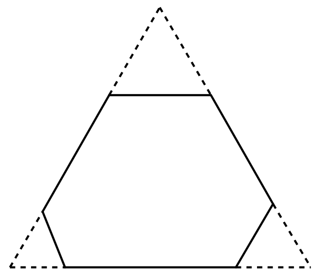
Remark. This question essentially asks for the number of partitions of the set {1, 2, 3, 4, 5, 6}. The answer is in fact the 6th *Bell number*.

14. If the binary representation of n is $\overline{a_k a_{k-1} a_{k-2} \cdots a_2 a_1}$, where each a_i is 0 or 1, then we have $\left[\frac{n}{2}\right] = \overline{a_k a_{k-1} a_{k-2} \ldots a_2}$ and $n - 2\left[\frac{n}{2}\right] = a_1$. It follows that

$$\begin{aligned} f(n) &= f(\overline{a_k a_{k-1} a_{k-2} \ldots a_2 a_1}) \\ &= f(\overline{a_k a_{k-1} a_{k-2} \ldots a_2}) + a_1 \\ &= f(\overline{a_k a_{k-1} a_{k-2} \ldots a_3}) + a_2 + a_1 \\ &= \cdots \\ &= f(a_k) + a_{k-1} + a_{k-2} + \cdots + a_2 + a_1 \\ &= a_k + a_{k-1} + a_{k-2} + \cdots + a_2 + a_1 \end{aligned}$$

which is number of 1's in the binary representation of n. For $0 \le n \le 2010$, the binary representation of n has at most 11 digits since $2^{10} < 2010 < 2^{11}$, and consists of at most ten 1's (as $11111111111_2 = 2047 > 2010$). Hence the maximum value of $f(n)$ is 10, as we can check, for instance, that $f(1023)=f(2^{10}-1)=f(1111111111_2)=10$.

15. We first observe that by extending three mutually non-adjacent sides of an equiangular hexagon, we get an equilateral triangle. From this we see that every equiangular hexagon is formed by removing three small equilateral triangles from a large equilateral triangle. Conversely, any such removal gives an equiangular hexagon.



It is thus easy to see that if the lengths of the consecutive sides of an equiangular hexagon are a, b, c, d, e, f, then we must have $a-d=-b+e=c-f$ (note that opposite sides form a pair and the lengths alternate in sign in the above equality; this also says that the sum of the lengths of three consecutive sides, which is also the side length of the equilateral triangle formed, may only take two different possible values, namely, either $a+b+c=c+d+e=e+f+a$ or $b+c+d=d+e+f=f+a+b$). Furthermore, since an equilateral triangle of unit length has area $\frac{\sqrt{3}}{4}$, we see that the equiangular hexagon formed from removing equilateral triangles of sides lengths a, c, e from a large equilateral triangle of side length n has area $\frac{\sqrt{3}}{4}(n^2-a^2-c^2-e^2)$, where $n=a+b+c$.

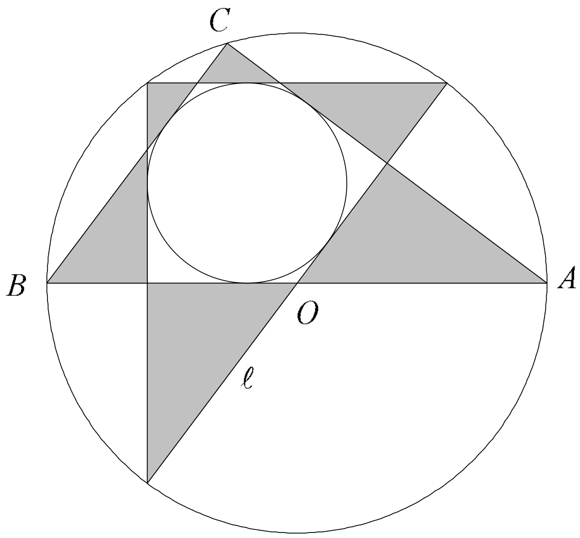
Now we must divide the lengths 6, 7, 8, 9, 10, 11 into three pairs with equal difference. Since the perimeter is 51, which is odd, the possible equal differences include 1 (the pairing is {6,7}, {8,9}, {10,11}) and 3 (the pairing is {6,9}, {7,10}, {8,11}).

In the first case, the two sides adjacent to 6 must be 9 and 11 (i.e. the two larger elements of the other two pairs, since 6 is the smaller in its pair), and the side lengths in order must be (6, 9, 10, 7, 8, 11). In this case the area of the hexagon is $\frac{\sqrt{3}}{4}(25^2-6^2-10^2-8^2)=\frac{425}{4}\sqrt{3}$. (Another expression would be $\frac{\sqrt{3}}{4}(26^2-9^2-7^2-11^2)$ which is the same.)

Similarly, in the second case, the two sides adjacent to 6 must be 10 and 11, and the side lengths in order must be (6, 10, 8, 9, 7, 11), and the area of the hexagon in this case is $\frac{\sqrt{3}}{4}(24^2-6^2-8^2-7^2)=\frac{427}{4}\sqrt{3}$.

Combining these two cases, we see that the answer is $\frac{425}{4}+\frac{427}{4}=213$.

16. First of all, note that a 18-24-30 triangle is a 6-time magnification of a 3-4-5 triangle, and so is right-angled. Thus the midpoint of the hypotenuse of each triangle is the centre of their common circumcircle, and the in-radius is $(18+24-30)\div 2=6$. Let one of the triangles be ΔABC where $\angle A<\angle B<\angle C=90°$, with circumcentre O.

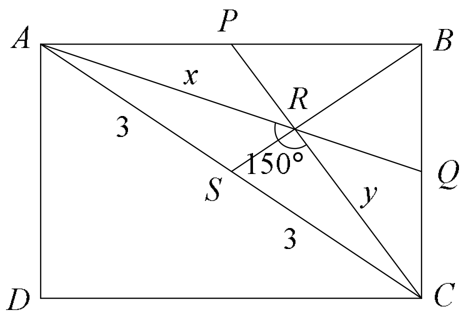


Let ℓ be the diameter of the circumcircle passing through the mid-point of AC. Since O is the mid-point of AB, ℓ is parallel to BC, and since $AC=24$, it is 12 units from BC. As the diameter of the inscribed circle is also 12 units, ℓ is tangent to the inscribed circle.

Therefore ℓ must be the hypotenuse of the other triangle. It is then easy to see that the regions which belong to exactly one of the two triangles (i.e. the six shaded triangular regions in the figure) are all similar to ΔABC.

The area of ΔABC is $18\times 24\div 2=216$. We focus on the three shaded regions at A, B and C. At A, the area of the shaded region is clearly $216\div 4=54$ (in fact, it is a 9-12-15 triangle). The region at B is a 6-8-10 triangle with area $6\times 8\div 2=24$. (To see this, we note that the line segment $BO=15$ is split into two parts, with the second part being of length 9 as can be seen by comparing with the 9-12-15 shaded triangle at A.) Similarly, the region at C is a 3-4-5 triangle with area $3\times 4\div 2=6$. It follows that the answer is $216-54-24-6=132$.

17. Let S be the mid-point of AC. Then R lies on BS with $BR:RS=2:1$ since R is the centroid of ΔABC. We have $BS=3$ (as S is the centre and AB is a diameter of the circumcircle of ΔABC) and so $RS=1$. Let $AR=x$ and $CR=y$. Since the area of ΔARC is $\frac{1}{2}xy\sin 150°=\frac{1}{4}xy$, the area of ΔABC is $\frac{3}{4}xy$ and hence the area of $ABCD$ is $\frac{3}{2}xy$.



To find xy, we first apply the cosine law in ΔARC, ΔARS and ΔCRS to get

$$\begin{aligned}36&=x^2+y^2-2xy\cos 150°\\&=(3^2+1^2-2\cdot 3\cdot 1\cdot\cos\angle RSA)+(3^2+1^2-2\cdot 3\cdot 1\cdot\cos\angle RSC)+\sqrt{3}xy.\\&=20+\sqrt{3}xy\end{aligned}$$

which gives (note that $\cos\angle RSA = -\cos\angle RSC$). It follows that $xy = \frac{16}{\sqrt{3}}$ and hence the answer is $\frac{3}{2}\left(\frac{16}{\sqrt{3}}\right) = 8\sqrt{3}$.

18. Upon expansion, we get $2(100x^2+260x+169)(5x^2+13x+8)=1$. If we let $u=5x^2+13x+8$, the equation becomes $2(20u+9)u=1$, which gives $u=\frac{1}{20}$ or $u=-\frac{1}{2}$, i.e. $5x^2+13x+\frac{159}{20}=0$ or $5x^2+13x+\frac{17}{2}=0$. The former gives two real roots with product $\frac{159}{100}$, and the latter gives two complex roots with product $\frac{17}{10}$. The two roots in the same equation must be paired up in order for $pq+rs$ to be real. It follows that the answer is $\frac{159}{100}+\frac{17}{10}=\frac{329}{100}$.

19. Since $a^2+3ab+b^2-2a-2b+4=(a-1)^2+(b-1)^2+3ab+2>0$ for all a, b, we see that $a\circ b$ is always well-defined and positive when a, b are positive. When a is positive, we have

$$a\circ 2 = \frac{\sqrt{a^2+6a+4-2a-4+4}}{2a+4} = \frac{\sqrt{(a+2)^2}}{2(a+2)} = \frac{1}{2}.$$

Hence $((\cdots((2010\circ 2009)\circ 2008)\circ\cdots\circ 2)\circ 1) = \frac{1}{2}\circ 1 = \frac{\sqrt{15}}{9}$.

Remark. Clearly, the crucial observation of this question is that $a\circ 2$ is a constant. What leads us to consider this? One way is to write the expression in the square root as a function of a, i.e. $a^2+(3b-2)a+(b^2-2b+4)$, with discriminant $(3b-2)^2-4(b^2-2b+4)=5b^2-4b-12$. By some trial and error we see that this is a perfect square when $b=2$.

20. Let $z=\sqrt[5]{x^3+20x}=\sqrt[3]{x^5-20x}$. Then we have $z^5=x^3+20x$ and $z^3=x^5-20x$. Adding, we have $z^5+z^3=x^5+x^3$, or $z^5-x^5+z^3-x^3=0$. Hence we have

$$0=(z-x)(z^4+z^3x+z^2x^2+zx^3+x^4+z^2+zx+x^2).$$

As x is non-zero, z is also non-zero. Thus we have $z^2+zx+x^2=\left(z-\frac{x}{2}\right)^2+\frac{3}{4}x^2>0$ and $z^4+z^3x+z^2x^2+zx^3+x^4=z^2\left(z+\frac{x}{2}\right)^2+x^2\left(x+\frac{z}{2}\right)^2+\frac{1}{2}z^2x^2>0$. Therefore the second factor on the right hand side of the above equation is positive. This forces $z=x$ and hence we get $x=\sqrt[5]{x^3+20x}$, or $0=x^5-x^3-20x=x(x^2+4)(x^2-5)$. The possible non-zero real values of x are thus $\pm\sqrt{5}$ and so the answer is –5.