International Mathematical Olympiad

Preliminary Selection Contest 2008 — Hong Kong

Outline of Solutions

Answers:

1. -143	**2.** 55	**3.** 27	**4.** 42
5. $14\sqrt{3}$	**6.** 270	**7.** 32	**8.** 1185
9. $\frac{7}{8}$	**10.** 660	**11.** 990	**12.** $\frac{2}{\sqrt{5}}$
13. $\pi+2$	**14.** $\frac{33}{8}$	**15.** 13	**16.** $2\sqrt{3}-3$
17. 27	**18.** $\frac{168}{295}$	**19.** -3	**20.** 4

Solutions:

1. Multiplying both sides by xy, we get $x^2+22x+290=26y-y^2$. Since $290=11^2+13^2$, this is equivalent to $(x+11)^2+(y-13)^2=0$. As x and y are real, we must have $x=-11$ and $y=13$, giving $xy=-143$.

2. Without loss of generality assume $a<b<c$. Let $a+b=x^2$, $a+c=y^2$ and $b+c=z^2$ for positive integers $x<y<z$. Then $a+b+c=\frac{1}{2}(x^2+y^2+z^2)$, so either x, y, z are all even or exactly one of them is even. Since $z^2=b+c<b+c+2a=x^2+y^2$, we have

$$z^2<x^2+y^2\le(z-2)^2+(z-1)^2,$$

which simplifies to $(z-1)(z-5)>0$. Hence z is at least 6. If $z=6$, the only possibility is (x, y) = (4, 5) in order that $z^2<x^2+y^2$, but this violates the parity constraint. It follows that z is at least 7, and the bound $z^2<x^2+y^2$ together with the parity constraint force (x, y) = (5, 6), and thus we get $\frac{1}{2}(5^2+6^2+7^2)=55$ as the smallest possible value of $a+b+c$, which corresponds to (a, b, c) = (6, 19, 30).

3. If one gets all four questions wrong, his score is 0; if one gets all four questions correct, his score may be 30, 40, 50 or 60. These give 5 possible scores.

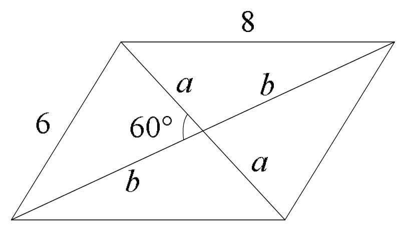
Now suppose one gets one, two or three questions correct. Then the score before the time bonus may be any integer from 1 to 9. With the time bonus of 1 to 4, there are 36 possible scores from 1 to 36. Of these, 30 is double-counted, 11, 22, 33, 13, 26, 17, 34, 19, 23, 29, 31 contain a prime factor greater than 10 and have to be discarded, and 25, 35 have their smallest prime factor greater than 4 and have to be discarded. The rest are possible scores. Hence there are $36-1-11-2=22$ possible scores in this case, and so the answer is $5+22=27$.

Remark. Another (perhaps easier) way of counting is to list the values of xy ($0\le x\le 10$, $0\le y\le 4$) and then count directly.

4. Note that the difference between $\overline{xy}$ and $\overline{yx}$ is $(10x+y)-(10y+x)=9(x-y)$, which is divisible by 9. It follows that $\overline{xy}$ is not equal to $\overline{yx}-3$, so from the equation we see that $x-y$ is at least 2. On the other hand, $x-y$ is at most 4, for otherwise $(x-y)!$ would exceed 100.

Rewrite the equation as $10x+y=(x-y)!(10y+x-3)$. If $x-y=2$, then eliminating x gives $11y+20=2(11y-1)$, which yields $y=2$ and hence an answer 42. On the other hand, setting $x-y=3$ and $x-y=4$ yield no integral solution, so that the answer is unique.

5. Let the two diagonals of the parallelogram have lengths $2a$ and $2b$ as shown. Then the cosine law gives



$$6^2=a^2+b^2-2ab\cos 60°=a^2+b^2-ab$$
$$8^2=a^2+b^2-2ab\cos 120°=a^2+b^2+ab$$

Solving gives $a^2+b^2=50$ and $ab=14$. Hence the parallelogram has area $2ab\sin 60°=14\sqrt{3}$.

6. Note that for non-negative real numbers a and b, we have $\left(\sqrt{a}-\sqrt{b}\right)^2\ge 0$ and hence $a+b\ge 2\sqrt{ab}$. It follows that $2008\ge 69x+54y\ge 2\sqrt{69x\cdot 54y}$ and hence

$$xy\le\left(\frac{2008}{2\sqrt{69\cdot 54}}\right)^2=\frac{1004^2}{69\cdot 54}\approx 270.5.$$

Thus xy is at most 270. This maximum is attainable when $x=15$ and $y=18$, as $69(15)+54(18)=2007$. Therefore the answer is 270.

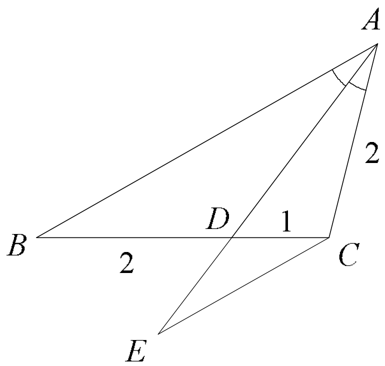
7. Note that $n^6+8=(n^2)^3+2^3$ is divisible by n^2+2. Hence n^2+2 must divide $206-8=198$. Among the factors of 198, we check that only five of them are of the form n^2+2 for positive integer *n*, namely, $3=1^2+2$, $6=2^2+2$, $11=3^2+2$, $18=4^2+2$, $66=8^2+2$ and $198=14^2+2$. It follows that the possible values of *n* are 1, 2, 3, 4, 8, 14 and the answer is $1+2+3+4+8+14=32$.

8. For positive integer *n*, let $f(n)$ denote the number of ending zeros of $n!$. Then $f(n)$ is equal to $\left[\frac{n}{5}\right]+\left[\frac{n}{5^2}\right]+\left[\frac{n}{5^3}\right]+\cdots$. It is non-decreasing and changes at and only at multiples of 5. Note also that $f(n)$ is slightly less than $\frac{n}{4}$, by taking its approximation with the greatest integer functions removed and the sum to infinity of the geometric series taken. Hence for $f(n)=57$, *n* is slightly greater than $57\times 4=228$. Hence we try $n=230$ and $n=235$ and get

$$f(230)=\left[\frac{230}{5}\right]+\left[\frac{230}{5^2}\right]+\left[\frac{230}{5^3}\right]=56 \text{ and } f(235)=\left[\frac{235}{5}\right]+\left[\frac{235}{5^2}\right]+\left[\frac{235}{5^3}\right]=57$$

respectively. It follows that $f(n)=57$ for and only for *n* = 235, 236, 237, 238 and 239. The answer is thus $235+236+237+238+239=1185$.

9. Produce *AD* to *E* so that *AB* // *CE*. Then $\angle CAE=\angle BAD=\angle CEA$ and so *CE* = *CA* = 2. As $\Delta ABD \sim \Delta ECD$, we have $\frac{AB}{EC}=\frac{BD}{CD}$, or $\frac{AB}{2}=\frac{2}{1}$, which gives *AB* = 4. Applying cosine law in ΔABC, we have $2^2=4^2+3^2-2(4)(3)\cos\angle ABC$ and hence $\cos\angle ABC=\frac{7}{8}$.



10. Note that $0.\dot{x}y\dot{z}=\frac{\overline{xyz}}{999}=\frac{\overline{xyz}}{3^3\times 37}$ and $\overline{xyz}$ may range from 001 to 999. Clearly each number from 1 to 999 that is relatively prime to 999 (i.e. not divisible by 3 and 37) gives rise to a possible value of the numerator. The number of such possible values is

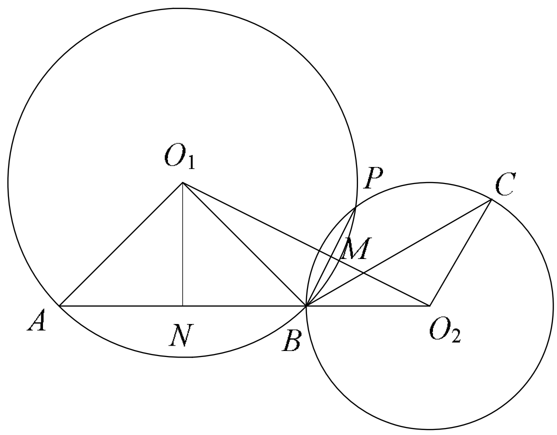
$$999-\left(\left[\frac{999}{3}\right]+\left[\frac{999}{37}\right]-\left[\frac{999}{3\times 37}\right]\right)=648.$$

Now it remains to count those numbers from 1 to 999 which are not relatively prime to 999 and which give rise to new possible values of numerators. These are precisely numbers divisible by 3^4 (we need not care about the factor 37 as 37^2 already exceeds 1000). There are 12 such numbers, namely, 81, 162, …, 972, each of which gives rise to a new possible numerator (they are 3, 6, …, 36 respectively). It follows that the answer is $648+12=660$.

11. The number of ways of selecting two gloves of the same colour is $C_2^x + C_2^y$ while that of of selecting two gloves of different colours is xy. We thus have $\frac{x(x-1)}{2}+\frac{y(y-1)}{2}=xy$, which simplifies to $(x-y)^2 = x+y$. Since $x=\frac{(x-y)+(x+y)}{2}=\frac{(x-y)+\sqrt{x-y}}{2}$, maximising x is equivalent to maximising $x-y$. As $(x-y)^2 = x+y \le 2008$, the maximum possible value of $x-y$ is 44. This corresponds to $x+y=44^2=1936$ and the maximum possible value of x being $\frac{44+1936}{2}=990$.

12. As shown in the figure, let O_1 and O_2 be the circumcentres of ΔABP and ΔBCP respectively, M be the mid-point of BP and N be the mid-point of AB. Since angle at centre is twice angle at circumference, we get $\angle AO_1B = 90°$ and the reflex $\angle BO_2C = 240°$. Hence

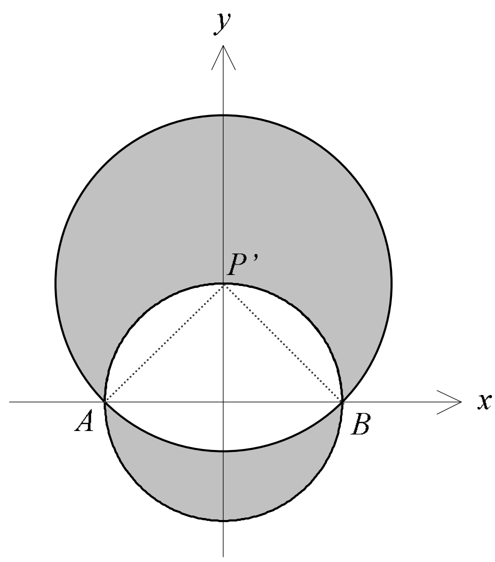
$$\angle CBO_2 = \frac{180° - (360° - 240°)}{2} = 30°$$



and so A, B, O_2 are collinear.

Note that $O_1N \perp AB$ and O_1, M, O_2 are collinear. Furthermore, we have $BP \perp O_1O_2$ and simple computation gives $O_1N = NB = BO_2 = 1$. Note also that $\Delta BMO_2 \sim \Delta O_1NO_2$, so $\frac{BM}{O_1N}=\frac{BO_2}{O_1O_2}$, or $\frac{BM}{1}=\frac{1}{\sqrt{1^2+2^2}}$, so that $BP = 2BM = \frac{2}{\sqrt{5}}$.

13. With a fixed P (0, t) where $0 \le t \le 1$, we can construct a circle with centre P and radius PA (which is the same as PB). Every point on the circumference of the circle is a possible position of C. In particular, when t takes the extreme values 0 and 1, we get two circles with centres (0, 0) and P' = (1, 1) respectively and with radius 1 and $\sqrt{2}$ respectively, and they provide the boundary for the red region. In the figure, the shaded region is the set of red points.



Note that $\angle AP'B = 90°$, so sector $P'AB$ has area $\frac{1}{4}\pi\left(\sqrt{2}\right)^2 = \frac{\pi}{2}$. Each remaining portion of the white enclosed region has area $\frac{1}{4}\pi(1)^2 - \frac{1}{2}(1)(1) = \frac{\pi}{4} - \frac{1}{2}$, so the entire white enclosed area has area $\frac{\pi}{2} + 2\left(\frac{\pi}{4} - \frac{1}{2}\right) = \pi - 1$. Consequently, the total area of the shaded region $\left[\pi\left(\sqrt{2}\right)^2 - (\pi - 1)\right] + \left[\pi(1)^2 - (\pi - 1)\right] = \pi + 2$.

14. Let $S = \frac{1^3}{3^1} + \frac{2^3}{3^2} + \frac{3^3}{3^3} + \frac{4^3}{3^4} + \cdots$. Then $\frac{S}{3} = \frac{1^3}{3^2} + \frac{2^3}{3^3} + \frac{3^3}{3^4} + \frac{4^3}{3^5} + \cdots$.

Subtracting the latter from the former, we have

$$\frac{2S}{3} = \frac{1}{3} + \frac{2^3 - 1^3}{3^2} + \frac{3^3 - 2^3}{3^3} + \frac{4^3 - 3^3}{3^4} + \cdots \qquad \ldots\ldots(1)$$

Dividing by 3, we have

$$\frac{2S}{9} = \frac{1}{9} + \frac{2^3 - 1^3}{3^3} + \frac{3^3 - 2^3}{3^4} + \frac{4^3 - 3^3}{3^5} + \cdots \qquad \ldots\ldots(2)$$

Subtracting (2) from (1), we get

$$\frac{4S}{9} = \frac{2}{9} + \frac{7}{9} + \frac{3^3 - 2\cdot 2^3 + 1^3}{3^3} + \frac{4^3 - 2\cdot 3^3 + 2^3}{3^4} + \cdots$$

Since $(n+1)^3 - 2n^3 + (n-1)^3 = 6n$, we actually have

$$\frac{4S}{9} = 1 + \frac{12}{3^3} + \frac{18}{3^4} + \frac{24}{3^5} + \cdots \qquad \ldots\ldots(3)$$

Dividing by 3, we get

$$\frac{4S}{27} = \frac{1}{3} + \frac{12}{3^4} + \frac{18}{3^5} + \frac{24}{3^6} + \cdots \qquad \ldots\ldots(4)$$

Subtracting (4) from (3) gives

$$\frac{8S}{27} = \frac{2}{3} + \frac{12}{27} + 6\left(\frac{1}{3^4} + \frac{1}{3^5} + \cdots\right) = \frac{10}{9} + 6\cdot\frac{\frac{1}{3^4}}{1 - \frac{1}{3}} = \frac{11}{9}.$$

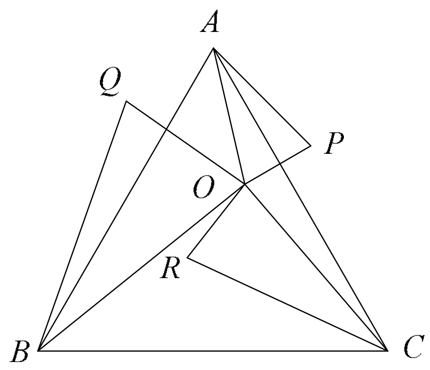
Solving gives $S = \frac{33}{8}$.

15. Note that $L = 2^4 \times 3^2 \times 5 \times 7 \times 11 \times 13 \times 17 \times 19$. Suppose $L \div n$ is a factor of L and is divisible by exactly 18 of the numbers 1, 2, …, 20. We say that a number m ($1 \le m \le 20$) is 'killed' if $L \div n$ is not divisible by m. Hence we are looking for choices of n which 'kill' exactly two of 1, 2, …, 20.

Note that if n is even, then 16 is 'killed'; if n is divisible by 4, then 8 and 16 are 'killed'; if n is divisible by 8, then more than two numbers are killed. Likewise, if n is divisible by 3, then 9 and 18 are 'killed'; if n is divisible by 9, then more than two numbers are 'killed'. On the other hand, the inclusion of the prime factor 5 in n 'kills' four numbers (5, 10, 15, 20), the inclusion of 7 'kills' two (7 and 14), while the inclusion of 11, 13, 17 or 19 'kills' one.

To 'kill' exactly two numbers, the possible values of n are therefore 4, 3, 7 or the product of exactly two of 2, 11, 13, 17 and 19. It follows that the answer is $3 + C_2^5 = 13$.

16. Note that OAP, OBQ and OCR are similar isosceles triangles with base angles $(180° - 30°) \div 2 = 75°$. Hence $\Delta AOB \sim \Delta POQ$ since $\frac{AO}{PO} = \frac{OB}{OQ}$ and $\angle AOB = \angle POQ$ (which follows from $\angle AOP = \angle BOQ$). Hence we have $\frac{AB}{PQ} = \frac{OB}{OQ}$, or $PQ = \frac{2OQ}{OB} = 4\cos 75°$. In the same way we see that QR and RP have the same length. Hence ΔPQR is indeed equilateral with area



$$\begin{aligned}\frac{\sqrt{3}}{4}(4\cos 75°)^2 &= 4\sqrt{3}\cos^2 75° \\ &= 2\sqrt{3}(1 + \cos 150°) \\ &= 2\sqrt{3} - 3\end{aligned}$$

Remark. It would be easy to guess the answer if one assumes O to be the centre of ΔABC.

17. Taking reciprocals, we have $6\frac{55}{72} = \frac{487}{72} > \frac{q}{p} > \frac{121}{18} = 6\frac{13}{18}$. Hence $q = 6p + r$ for some $0 < r < p$. Now subtracting 6 from each term, we get $\frac{55}{72} > \frac{r}{p} > \frac{13}{18}$ and hence $\frac{72}{55}r < p < \frac{18}{13}r$. To minimise q we should therefore look for the smallest r for which there exists an integer p between $\frac{72}{55}r$ and $\frac{18}{13}r$. It is easy to check that the smallest such r is 3. This corresponds to $p = 4$ and $q = 6 \times 4 + 3 = 27$.

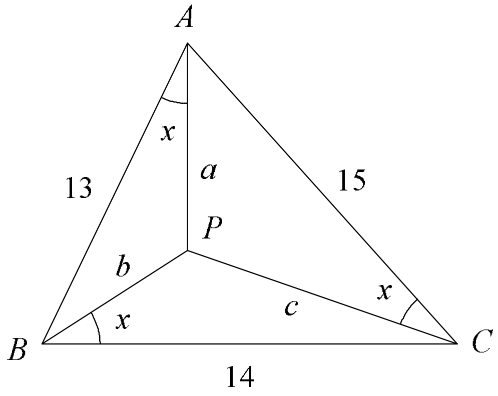
18. The semi-perimeter of ΔABC is $(13+14+15)\div 2=21$ and hence ΔABC has area

$$\sqrt{21(21-13)(21-14)(21-15)}=84$$

by Heron's formula. Let $\angle PAB = \angle PBC = \angle PCA = x$, $PA=a$, $PB=b$ and $PC=c$. Then we also have

$$84=\frac{1}{2}(13)(a)\sin x+\frac{1}{2}(14)(b)\sin x+\frac{1}{2}(15)(c)\sin x$$

and hence $(13a+14b+15c)\sin x=168$.



Applying cosine law in ΔPAB, ΔPBC and ΔPCA respectively, we have

$$b^2=a^2+13^2-2(a)(13)\cos x$$
$$c^2=b^2+14^2-2(b)(14)\cos x$$
$$a^2=c^2+15^2-2(c)(15)\cos x$$

Adding up the three equations and simplifying, we get

$$2(13a+14b+15c)\cos x=13^2+14^2+15^2.$$

It follows that $\tan x=\frac{\sin x}{\cos x}=\frac{168}{13a+14b+15c}\cdot\frac{2(13a+14b+15c)}{13^2+14^2+15^2}=\frac{168}{295}$.

19. Set $x=\sin 50°$. Then the given equation becomes $\sqrt{9-8x}=a+\frac{b}{x}$. Squaring both sides and rearranging terms, we get the cubic equation $8x^3+(a^2-9)x^2+2abx+b^2=0$.

This cubic equation reminds us of the triple angle formula $\sin 3\theta=3\sin\theta-4\sin^3\theta$, which gives $\frac{1}{2}=\sin 150°=3x-4x^3$ and hence $8x^3-6x+1=0$. Reversing the steps in the previous paragraph, we get $(9-8x)x^2=(3x-1)^2$ and hence $\sqrt{9-8x}=3-\frac{1}{x}$, noting that $x>\frac{1}{3}$. It follows that we may take $(a,b)=(3,-1)$ which gives the answer $ab=-3$.

Finally, we show that the choices for a, b are unique. Suppose on the contrary that $a+b\csc 50°=a'+b'\csc 50°$ where $a\neq a'$. Then rearranging terms gives $x=\sin 50°=\frac{b'-b}{a-a'}$, which is rational. However, it is easy to check that ± 1, $\pm\frac{1}{2}$, $\pm\frac{1}{4}$ and $\pm\frac{1}{8}$ are not roots of $8x^3-6x+1=0$, so the equation has no rational root, a contradiction.

20. Observe that for a prime power p^r and $0 < i < p^r$, the number

$$C_i^{p^r} = \frac{p^r(p^r-1)\cdots(p^r-i+1)}{i(i-1)\cdots(3)(2)(1)}$$

is always divisible by p because the power of p in the numerator is strictly greater than that in the denominator (as the number of multiples of p^r is 1 in the numerator and 0 in the denominator, and the number of multiples of lower powers of p in the numerator is at least equal to that in the denominator). In order words, we have $(1+x)^{p^r} \equiv 1+x^{p^r} \pmod{p}$ and so

$$\begin{aligned}(1+x)^{38} &= (1+x)^{27}(1+x)^9(1+x)^2 \\ &\equiv (1+x^{27})(1+x^9)(1+2x+x^2) \\ &= (1+x^9+x^{27}+x^{36})(1+2x+x^2) \qquad \pmod{3}\end{aligned}$$

Upon expanding the two parentheses in the last row, we get 12 terms, 8 of which have coefficient 1 and 4 have coefficient 2. It follows that $N_1 - N_2 = 8 - 4 = 4$.

Remark. A more intuitive approach would be to construct a Pascal triangle (taken modulo 3). One could try to observe some pattern in the process (e.g. noting that both Row 3 and Row 9 are of the form 10…01) to reduce the amount of brute force needed.