International Mathematical Olympiad

Preliminary Selection Contest 2004 — Hong Kong

Outline of Solutions

Answers:

1.	8	**2.**	10	**3.**	$\frac{3}{2}-N$	**4.**	49894
5.	$\frac{1}{2}-\frac{1}{2004!}$	**6.**	$\frac{4011}{4010}$	**7.**	$63°$	**8.**	171
9.	5	**10.**	103	**11.**	$\frac{3007}{2}$	**12.**	280616
13.	$p+2$	**14.**	35	**15.**	603	**16.**	2475
17.	7	**18.**	540	**19.**	$\sqrt{3}:2$	**20.**	518656

Solutions:

1. A total of $C_3^8 = 56$ triangles can be formed by joining any three vertices of the cuboid. Among these, if any two vertices of a triangle are adjacent vertices of the cuboid, the triangle is right-angled. Otherwise, it is acute angled. To see this latter statement, note that if the dimensions of the cuboid are $p \times q \times r$, then we can find from the cosine law that the cosines of the angles of such a triangle will be equal to $\frac{p^2}{(p^2+q^2)(p^2+r^2)}$, $\frac{q^2}{(q^2+p^2)(q^2+r^2)}$ and $\frac{r^2}{(r^2+p^2)(r^2+q^2)}$, which are all positive.

   For each fixed vertex (say A), we can form 6 triangles which are right-angled at A (two on each of the three faces incident to A). Thus the answer is $56-6\times 8=8$.

2. Let the total weight of the stones be 100. Then the weight of the three heaviest stones is 35 and that of the three lightest stones is $(100-35)\times\frac{5}{13}=25$. The remaining $N-6$ stones, of total weight $100-35-25=40$, has average weight between $\frac{25}{3}$ and $\frac{35}{3}$. Since $40\div\frac{35}{3}>3$ and

$40 \div \frac{25}{3} < 5$, we must have $N-6=4$, from which the answer $N=10$ follows.

3. From the first equation, we have $2y = N+2-[x]$, which is an integer. Hence y is either an integer or midway between two integers. The same is true for x by looking at the second equation. Hence, either $[x]=x$ or $[x]=x-\frac{1}{2}$, and either $[y]=y$ or $[y]=y-\frac{1}{2}$.

   If x and y are both integers, $[x]=x$ and $[y]=y$. Solving the equations, we get $x=\frac{4}{3}-N$ and $y=N+\frac{1}{3}$ which are not consistent with our original assumptions.

   If x is an integer and y is midway between two integers, $[x]=x$ and $[y]=y-\frac{1}{2}$. Solving the equations, we have $x=\frac{5}{3}-N$ and $y=N+\frac{1}{6}$ which are not consistent with our original assumptions.

   Similarly, if x is midway between two integers and y is an integer, we have $[x]=x-\frac{1}{2}$ and $[y]=y$. Solving the equations, we have $x=\frac{7}{6}-N$ and $y=N+\frac{2}{3}$ which are again not consistent with our original assumptions.

   Finally, if x and y are both midway between two integers, we have $[x]=x-\frac{1}{2}$ and $[y]=y-\frac{1}{2}$. Solving the equations, we have $x=\frac{3}{2}-N$ and $y=N+\frac{1}{2}$. This gives the correct answer.

4. Let the answer be $\overline{abcba}$. Note that

$$\overline{abcba} = 10001a + 1010b + 100c = 101(99a+10b+c) + 2a - c$$

   For the number to be divisible by 101, we must have $2a-c=0$. For the number to be largest, we may take $a=4$, $c=8$ and $b=9$. This gives the answer is 49894.

5. Note that

$$\frac{k+2}{k!+(k+1)!+(k+2)!} = \frac{1}{k!(k+2)} = \frac{k+1}{(k+2)!} = \frac{1}{(k+1)!} - \frac{1}{(k+2)!}.$$

   Hence

$$\frac{3}{1!+2!+3!}+\frac{4}{2!+3!+4!}+\cdots+\frac{2004}{2002!+2003!+2004!}$$
$$=\left(\frac{1}{2!}-\frac{1}{3!}\right)+\left(\frac{1}{3!}-\frac{1}{4!}\right)+\cdots+\left(\frac{1}{2003!}-\frac{1}{2004!}\right)$$
$$=\frac{1}{2}-\frac{1}{2004!}$$

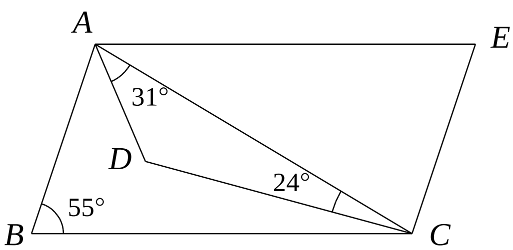
6. Let $x=a+b$, $y=b+c$ and $z=c+a$. Then

$$\frac{2004}{2005}=\frac{(a-b)(b-c)(c-a)}{(a+b)(b+c)(c+a)}=\frac{(z-y)(y-x)(x-z)}{xyz}$$

and hence

$$\frac{a}{a+b}+\frac{b}{b+c}+\frac{c}{c+a}=\frac{x-y+z}{2x}+\frac{y-z+x}{2y}+\frac{z-x+y}{2z}$$
$$=\left(\frac{1}{2}-\frac{y-z}{2x}\right)+\left(\frac{1}{2}-\frac{z-x}{2y}\right)+\left(\frac{1}{2}-\frac{x-y}{2z}\right)$$
$$=\frac{3}{2}-\frac{1}{2}\left[\frac{(z-y)(y-x)(x-z)}{xyz}\right]$$
$$=\frac{3}{2}-\frac{1}{2}\left(\frac{2004}{2005}\right)$$
$$=\frac{4011}{4010}$$

7. Draw E such that $ABCE$ is a parallelogram. Since $\angle AEC = \angle ABC = 55°$ and $\angle ADC = 180° - 31° - 24° = 125°$, we have $\angle AEC + \angle ADC = 180°$ and thus $ADCE$ is a cyclic quadrilateral. Now $EC = AB = DC$, so $\angle CDE = \angle CED = \angle CAD = 31°$. Considering ΔCDE, we have $\angle ACE = 180° - 31° - 31° - 24° = 94°$. It follows that $\angle DAB = \angle BAC - 31° = \angle ACE - 31° = 94° - 31° = 63°$.



(Alternatively, instead of drawing the point E, one can reflect B across AC and proceed in essentially the same way.)

8. We have

$$\begin{aligned}999973 &= 1000000-27\\ &= 100^3-3^3\\ &= (100-3)(100^2+100\times 3+3^2)\\ &= 97\times 10309\\ &= 97\times 13\times 793\\ &= 97\times 13^2\times 61\end{aligned}$$

and so the answer is $97+13+61=171$.

9. We first note that the five primes 5, 11, 17, 23, 29 satisfy the conditions. So *n* is at least 5.

Next we show that *n* cannot be greater than 5. Suppose there are six primes satisfying the above conditions. Let *a* be the smallest of the six primes and let *d* be the common difference of the resulting arithmetic sequence. Then *d* must be even, for if *d* is odd then exactly three of the six primes are even, which is not possible. Similarly, *d* must be divisible by 3, for otherwise exactly 2 of the six primes are divisible by 3, which is not possible.

Moreover, if *d* is not divisible by 5, then at least one of the six primes must be divisible by 5. Therefore 5 is one of the primes picked. But we have shown that *d* is divisible by 6, so 5 is the smallest among the six primes. But then the largest of the six primes, $5+5d$, is also divisible by 5 and is larger than 5. This is absurd.

Hence *d* is divisible by 2, 3 and 5, hence divisible by 30. So the largest of the 6 primes, which is $a+5d$, must be larger than 150, a contradiction. It follows that the answer is 5.

10. By setting $p(0)=1$, we may write $S = p(000)+p(001)+p(002)+\cdots+p(999)-p(000)$. Since we are now computing the product of non-zero digits only, we may change all the 0's to 1's, i.e. $S = p(111)+p(111)+p(112)+\cdots+p(999)-p(111)$. Each term is the product of three numbers, and each multiplicand runs through 1, 1, 2, 3, 4, 5, 6, 7, 8, 9 (note that 1 occurs twice as all 0's have been changed to 1's). Hence we see that

$$\begin{aligned}S &= (1+1+2+3+\cdots+9)\times(1+1+2+3+\cdots+9)\times(1+1+2+3+\cdots+9)-1\\ &= 46^3-1\\ &= (46-1)\times(46^2+46+1)\\ &= 3^2\times 5\times 2163\\ &= 3^2\times 5\times 3\times 7\times 103\end{aligned}$$

It follows that the answer is 103.

11. Using $\frac{1}{k(k+1)}=\frac{1}{k}-\frac{1}{k+1}$, we have

$$\begin{aligned}A&=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{2003}-\frac{1}{2004}\\&=\left(1+\frac{1}{2}+\cdots+\frac{1}{2004}\right)-2\left(\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2004}\right)\\&=\left(1+\frac{1}{2}+\cdots+\frac{1}{2004}\right)-\left(1+\frac{1}{2}+\cdots+\frac{1}{1002}\right)\\&=\frac{1}{1003}+\frac{1}{1004}+\cdots+\frac{1}{2004}\end{aligned}$$

Consequently,

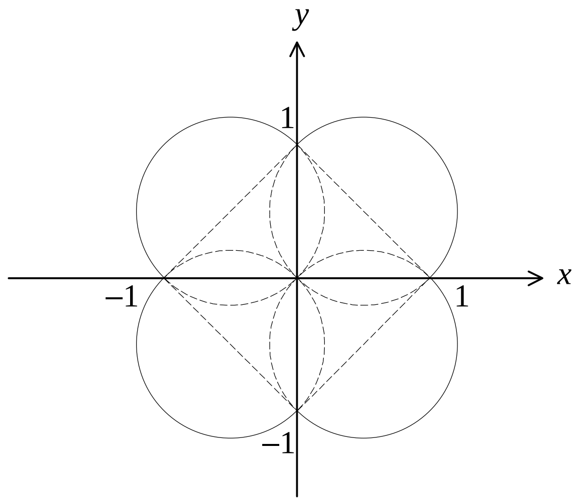
$$\begin{aligned}2A&=\left(\frac{1}{1003}+\frac{1}{2004}\right)+\left(\frac{1}{1004}+\frac{1}{2003}\right)+\cdots+\left(\frac{1}{2004}+\frac{1}{1003}\right)\\&=\frac{3007}{1003\times2004}+\frac{3007}{1004\times2003}+\cdots+\frac{3007}{2004\times1003}\\&=3007B\end{aligned}$$

It follows that $\frac{A}{B}=\frac{3007}{2}$.

12. If $a\times b$ is odd, both digits are odd and we have $5\times5=25$ choices. If it is even, we have $9\times9-5\times5=56$ choices (note that both digits cannot be zero). The same is true for the quantities $c\times d$ and $e\times f$.

Now, for condition (b) to hold, either all three quantities $a\times b$, $c\times d$ and $e\times f$ are even, or exactly two of them are odd. Hence the answer is $56^3+3\times56\times25^2=280616$.

13. Observe that the graph is symmetric about the x-axis and the y-axis. Hence we need only consider the first quadrant. In the first quadrant, the equation of the graph can be written as $x^2+y^2=x+y$, or $\left(x-\frac{1}{2}\right)^2+\left(y-\frac{1}{2}\right)^2=\frac{1}{2}$, which is a circle passing through (0, 0), (1, 0) and (0, 1). By symmetry, the whole graph can be constructed as shown.



Now the area bounded by the curve can be thought of as a square of side length $\sqrt{2}$ plus four semi-circles of diameter $\sqrt{2}$. Its area is

$$\left(\sqrt{2}\right)^2 + 4\cdot\frac{1}{2}\cdot p\left(\frac{\sqrt{2}}{2}\right)^2 = p + 2.$$

14. Using the identity

$$a^3+b^3+c^3-3abc=\frac{1}{2}(a+b+c)\left[(a-b)^2+(b-c)^2+(c-a)^2\right],$$

we have

$$\begin{aligned} m^3+n^3+99mn-33^3 &= m^3+n^3+(-33)^3-3mn(-33) \\ &= \frac{1}{2}(m+n-33)[(m-n)^2+(m+33)^2+(n+33)^2] \end{aligned}$$

For this expression to be equal to 0, we either have $m+n=33$ or $m=n=-33$. The latter gives one solution (–33, –33) while the former gives the 34 solutions (0, 33), (1, 32), …, (33, 0). Hence the answer is 35.

15. Let $f(n)$ be the number of significant figures when 2^{-n} is written in decimal notation. Then $f(n)=f(n+1)$ when n is 'lucky', and $f(n)+1=f(n+1)$ otherwise. Now $f(2)=2$, and we want to compute $f(2004)$. We first note that

$$2^{-2004}=\frac{5^{2004}}{10^{2004}},$$

so $f(2004)$ is equal to the number of digits of 5^{2004}. Now

$$\log 5^{2004}=2004(1-\log 2)\approx 2004\times(0.7-0.001)=1402.8-2.004=1400.796,$$

so 5^{2004} has 1401 digits. It follows that the number of lucky numbers less than 2004 is equal to $2004-1401=603$.

16. Let *n* be such a number. Since 3*n* is divisible by 3, the sum of the digits of 3*n* is divisible by 3. But the sum of the digits of 3*n* is the same as the sum of the digits of *n*, so the sum of digits of *n* is divisible by 3, i.e. *n* is divisible by 3. As a result, 3*n* is divisible by 9, so the sum of digits of 3*n* is divisible by 9. Again, the sum of the digits of 3*n* is the same as the sum of the digits of *n*, so the sum of digits of *n* is divisible by 9, i.e. *n* is divisible by 9.

Let $n=\overline{abcd}$. Since n is to be divisible by both 9 and 11, $a+b+c+d$ is divisible by 9 and $(a+c)-(b+d)$ is divisible by 11. Considering the parities of $a+c$ and $b+d$ we see that $a+b+c+d$ has to be equal to 18 with $a+c=b+d=9$.

The rest is largely trial and error. Noting that a can be no larger than 3, we have the possibilities (a, c) = (1, 8); (2, 7) or (3, 6). Considering the digits, the only possibilities for n

are 1287, 1386, 1485, 2079, 2475, 2574, 3465, 3762 and 3861. Among these, we find that only $n = 2475$ works as $3n = 7425$ in this case.

17. Note that $b = a(10^n + 1)$, so $\frac{b}{a^2} = \frac{10^n + 1}{a}$. Let this be an integer d. Noting that $10^{n-1} \le a < 10^n$ and $n > 1$, we must have $1 < d < 11$. Since $10^n + 1$ is not divisible by 2, 3 and 5, the only possible value of d is 7. Indeed, when $a = 143$, we have $b = 143143$ and $d = 7$.

18. By the AM-GM inequality, $9\tan^2 x° + \cot^2 x° \ge 2\sqrt{(9\tan^2 x°)(\cot^2 x°)} = 6$. It follows that the minimum value of the right hand side is 1. On the other hand, the maximum value of the left hand side is 1. For equality to hold, both sides must be equal to 1, and we must have $9\tan^2 x° = \cot^2 x°$ (which implies $\tan x° = \pm\frac{1}{\sqrt{3}}$), $\cos 12x° = 1$ and $\sin 3x° = -1$.

For $\tan x° = \pm\frac{1}{\sqrt{3}}$, the solutions are x = 30, 150, 210, 330.

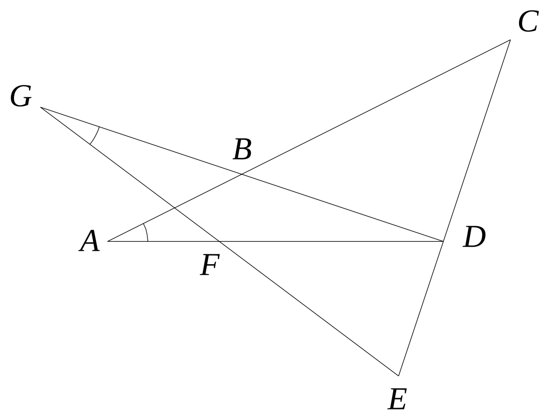
For $\cos 12x° = 1$, the solutions are x = 0, 30, 60, …, 330, 360.

For $\sin 3x° = -1$, the solutions are x = 90, 210, 330.

Therefore the equation has solutions x = 210, 330 and so the answer is $210 + 330 = 540$.

19. Let [DEF] = 2.

Since $CD : DE = 3 : 2$, [FCD] = 3.



Since $AB : BC = 1 : 2$, [FBD] = 1.

Since $EF = FG$, [GFD] = [DEF] = 2.

So [GBF] = 1 = [FBD], i.e. $GB = BD$.

Together with $GF = FE$, $BF \parallel CE$.

Hence $AF : FD = AB : BC = 1 : 2$.

Let $GB = BD = x$, $AF = y$, $FD = 2y$.

Since $\angle CAD = \angle EGD$, $GAFB$ is a cyclic quadrilateral.

Thus $DB \times DG = DF \times DA$, i.e. $x(2x) = 2y(3y)$.

As a result we have $x : y = \sqrt{3}$.

It follows that $BD : DF = x : 2y = \sqrt{3} : 2$.

20. The key observation is that $f(n)=n-g(n)$, where $g(n)$ is the number of 1's in the binary representation of *n*. To see this, it suffices to check that for non-negative integers $a_1<a_2<\cdots<a_k$ and $n=2^{a_1}+2^{a_2}+\cdots+2^{a_k}$, we have

$$\left[\frac{n}{2}\right]+\left[\frac{n}{2^2}\right]+\cdots+\left[\frac{n}{2^{a_k}}\right]=n-k\,.$$

Such checking is straightforward.

Next we try to compute $g(0)+g(1)+\cdots+g(1023)$. Note that the binary representations of 0 and 1023 are respectively 0000000000 and 1111111111, so as *n* runs through 0 to 1023, $g(n)$ is equal to 5 'on average'. Since $g(0)=0$, we have

$$g(1)+\cdots+g(1023)=5\times 1024=5120\,.$$

Now we have

$$\begin{aligned} f(1)+f(2)+\cdots+f(1023) &= (1+2+\cdots+1023)-\left[g(1)+g(2)+\cdots+g(1023)\right] \\ &= \frac{1023\times 1024}{2}-5120 \\ &= 512\times(1023-10) \\ &= 518656 \end{aligned}$$